Laplace Transforms

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Presentation Outline

1. **Introduction**
2. **Building the Bridges: time-domain and s-domain**
   - Laplace Transform: properties
3. **Transfer Functions**
4. **Laplace Transform Interpretations**
   - Laplace Transform as a "Generalized Fourier Transform"
   - Laplace Transform as a "Generalized Power Series"

Section 1

Introduction

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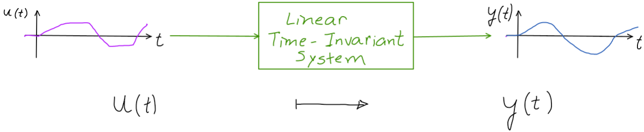
Long answer: you will be able to interpret the responses of Linear and Time Invariant (LTI) dynamic systems from another very powerful perspective, improving your understanding about what is happening and how you can change it (a.k.a how to design automatic controllers).

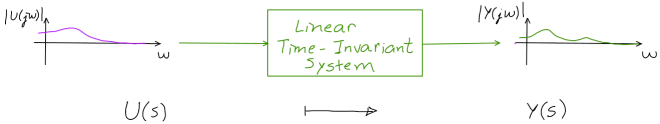
Caveat: it will take some time to really learn this subject...

LTI responses – Time-domain



- The LTI system response $y(t)$ to an input $u(t)$ can be seen as a *mapping* from the space $\mathcal{U}$ of input signals to the space $\mathcal{Y}$ of output signals.
- A signal can be described as a function of time, such as $u(t)$ and $y(t)$, and we say they are represented in the **time-domain**.

LTI responses – Frequency-domain



- Signals can also be represented in the **frequency-domain**. We do this by working with coefficients that describe how a signal is formed by a linear combination of basic oscillatory components. In this case, $U(s)$ encodes the coefficients for the input signal $u(t)$, and $Y(s)$ encodes the coefficients for the output signal $y(t)$.

Expression of the Output I

- Consider a system described by the following equations, with the input-output relations highlighted:

$$\begin{cases} \dot{x} = -x + u(t), \\ x(0) = 0, \\ y(t) = x. \end{cases} \implies \begin{matrix} y(t) = \int_0^t e^{-(t-\tau)} u(\tau) d\tau. \\ \text{or} \\ Y(s) = \frac{1}{s+1} U(s). \end{matrix}$$

- Using the second expression (frequency-domain) on the right it is easier to see what will happen if the input is a fast oscillatory signal such that the complex frequency $s \to \infty$: the output will go to zero. It is much harder to see this using the convolution integral in the first expression.

Expression of the Output II

- In the **time-domain**, considering zero initial conditions, we have that

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

where (A, B, C, D) are matrices used in the state-space representation of the system, with $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ (m scalar inputs and p scalar outputs).

- In the **frequency-domain**, considering zero initial conditions, we have that

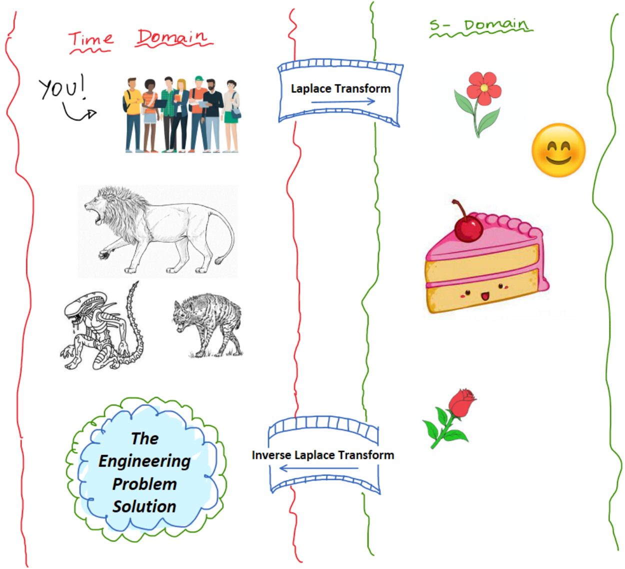
$$Y(s) = G(s)U(s)$$

where $G(s) \in \mathbb{C}^{p\times m}$ is a matrix of rational functions of the complex variable $s \in \mathbb{C}$.

Expression of the Output III

- Therefore we can see that we are trading the complexity of computing a **convolution integral** in the time-domain by the simplicity of performing a simple **multiplication of rational functions** in the frequency-domain.
- At the same time, we are abandoning the well-known time-domain, and starting to work in a much more abstract setting: the complex frequency $s-$domain.
- To do this, we first need to build the bridges that interconnect these two lands: the **time-domain** and the **frequency-domain**.

The Laplace Transform Way



Section 2

Building the Bridges: time-domain and s-domain

The Laplace Transform I

- The idea of a "Transform" is to transform the *representation* of the information from a domain to another domain. But the information itself should be preserved in this process.

The Laplace Transform II

- One way of doing this is to use integral transforms, where we compute an integral of the signal $u(t)$ depending on a variable "parameter" s, such that the result becomes dependent on this parameter s:

Laplace Transform:

$$U(s) = \int_{0^-}^{\infty} u(t)e^{-st}dt = \mathcal{L}\left\{u(t)\right\}$$

where 0^- represents the limit from the left in the time axis to capture any discontinuities at $t = 0$. Notice that:

1. All values of $u(t)$, with $t \geq 0$, are necessary to produce just one single value of $U(s)$.
2. The parameter s is a complex number: $s = \sigma + i\omega$, and therefore the result is a complex number too: $U(s) = \mathrm{Re}\left\{U(s)\right\} + i\,\mathrm{Im}\left\{U(s)\right\}$.
3. We are assuming that this improper integral converges for the chosen values of the parameter s.

The Laplace Transform III

- Let's do some examples (all functions are such that $u(t) = 0$ for $t < 0$):

1. $$u(t) = c \quad \Leftrightarrow \quad U(s) = \frac{c}{s}.$$

2. $$u(t) = t \quad \Leftrightarrow \quad U(s) = \frac{1}{s^2}.$$

3. $$u(t) = e^{-\lambda t} \quad \Leftrightarrow \quad U(s) = \frac{1}{s + \lambda}.$$

4. $$u(t) = \sin(\omega_0 t) \quad \Leftrightarrow \quad U(s) = \frac{\omega_0}{s^2 + \omega_0^2}.$$

5. $$u(t) = e^{-\alpha t}\cos(\omega_0 t) \quad \Leftrightarrow \quad U(s) = \frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}.$$

The Laplace Transform IV

- In practice, we consult a **Table of Laplace Transforms** like this one to obtain the expressions for a given function of time $u(t)$.
- In addition, we also rely on a series of very nice properties of the Laplace Transform, as we will see next.

Subsection 1

Laplace Transform: properties

Linearity

$$\mathcal{L}\{a_1u_1(t) + a_2u_2(t)\} = a_1\mathcal{L}\{u_1(t)\} + a_2\mathcal{L}\{u_2(t)\},$$
$$= a_1U_1(s) + a_2U_2(s).$$

with $a_1, a_2 \in \mathbb{R}$.

- With this property we can factor the components from a given signal in the frequency-domain, and easily see what will be the corresponding components in the time domain:

  Example:

$$U(s) = \frac{3s+5}{s^2+4s+3} = \underbrace{\frac{1}{s+1}}_{U_1(s)} + 2\underbrace{\frac{1}{s+3}}_{U_2(s)}$$

$$u_1(t) = e^{-t},$$
$$u_2(t) = 2e^{-3t},$$
$$u(t) = e^{-t} + 2e^{-3t}.$$

Laplace Transform: Time Derivation

$$\mathcal{L}\left\{\frac{du(t)}{dt}\right\} = s\mathcal{L}\left\{u(t)\right\} - u(0^-),$$
$$= sU(s) - u(0^-).$$

with $u(0^-)$ the initial value of $u(t)$, considering the limit from the left $\lim_{t\to 0^-} u(t)$ to capture any discontinuity at $t = 0$.

- Notice that, if the initial value $u(0^-) = 0$, to differentiate a function in the time-domain is equivalent to multiply its Laplace transform by s in the frequency-domain:

$$s \equiv \frac{d}{dt}(\cdot)$$

Laplace Transform: Integration

$$\mathcal{L}\left\{\int_{0^-}^{t} u(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\left\{u(t)\right\},$$
$$= \frac{1}{s}U(s).$$

- This means that to integrate a signal in the time-domain is equivalent to multiply its Laplace Transform by $\frac{1}{s}$ in the frequency-domain:

$$\frac{1}{s} \equiv \int_{0}^{t}(\cdot)d\tau$$

Frequency Shifting

$$\mathcal{L}\left\{e^{-at}u(t)\right\} = U(s+a).$$

- This is immediate from the definition

$$\mathcal{L}\left\{e^{-at}u(t)\right\} = \int_{0^-}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0^-}^{\infty} u(t)e^{-(s+a)t}dt = U(s+a),$$

and it will be very useful later.

Section 3

Transfer Functions

Transfer Functions – Motivation I

- Suppose that we have the following LTI system described by the set of differential equations

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = ax_1 + bx_2 + cu,$$

where $u \equiv u(t)$ is the input, and $x_1 \equiv x_1(t)$ and $x_2 \equiv x_2(t)$ are state variables, and by the algebraic output equation

$$y = x_1.$$

Transfer Functions – Motivation II

- We can use the Laplace Transform (and its properties) to write that

$$
\begin{aligned}
\mathcal{L}\{\dot{x}_1\} = \mathcal{L}\{x_2\} \quad & \Rightarrow sX_1 - x_1(0) = X_2, \\
\mathcal{L}\{\dot{x}_2\} = \mathcal{L}\{ax_1 + bx_2 + cu\} \quad & \Rightarrow sX_2 - x_2(0) = aX_1 + bX_2 + cU, \\
\mathcal{L}\{y\} = \mathcal{L}\{x_1\} \quad & \Rightarrow Y = X_1.
\end{aligned}
$$

where $U \equiv U(s)$, $X_1 \equiv X_1(s)$, $X_2 \equiv X_2(s)$, and $Y \equiv Y(s)$.
Notice that the differential equations were transformed into algebraic equations in the variable s. If we assume **zero initial conditions**, then

$$
\begin{aligned}
sX_1 &= X_2, \\
sX_2 &= aX_1 + bX_2 + cU, \\
Y &= X_1.
\end{aligned}
$$

Transfer Functions – Motivation III

- Now we can manipulate the previous algebraic relations to get

$$
\begin{aligned}
sX1 &= X_2, \\
sX_2 &= aX_1 + bX_2 + cU, \quad \Rightarrow \quad s(sX1) = aX_1 + b(sX1) + cU, \\
X_1 &= \frac{c}{s^2 - bs - a} U, \\
Y &= X_1, \quad \Rightarrow \quad Y(s) = \underbrace{\left[\frac{c}{s^2 - bs - a} \right]}_{G(s)} U(s).
\end{aligned}
$$

Transfer Functions – Motivation IV

- This means that we can determine the output of the system due to any input by knowing the so-called **Transfer Function**

$$G(s) = \frac{c}{s^2 - bs - a},$$

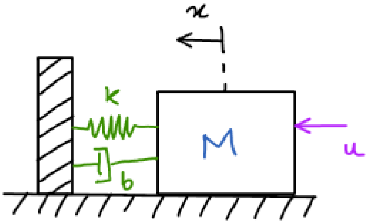
since $Y(s) = G(s)U(s)$.

- If we are interested in $y(t)$, we can compute the inverse Laplace Transform of the resulting $Y(s)$:

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{G(s)U(s)\right\}.$$

Transfer Functions – Motivation V

- Example:
  - Consider the mass-spring-damper system below, where x is the block position, M is the block mass, k is the spring elastic constant, b_{f} is the damper viscous friction coefficient, and u is the applied force:



$$M\ddot{x} = u - kx - b_{\mathrm{f}}\dot{x},$$

Transfer Functions – Motivation VI

- By defining the first state variable to be the position $x_1 = x$, and the second one to be the block's speed $x_2 = \dot{x} = \dot{x}_1$, and the position of the block as the output (the signal in which we are interested), we have that:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{k}{M}x_1 - \frac{b_{\mathrm{f}}}{M}x_2 + \frac{1}{M}u,$$
$$y = x_1,$$

which has exactly the same structure as before, with $c = \frac{1}{M}$, $a = -\frac{k}{M}$, and $b = -\frac{b_{\mathrm{f}}}{M}$, and therefore we know that the Transfer Function will be:

$$G(s) = \frac{\frac{1}{M}}{s^2 + \frac{b_{\mathrm{f}}}{M}s + \frac{k}{M}}.$$

Transfer Functions – Motivation VII

- Suppose $b_{\text{f}} = 0.5\,[\text{N/m/s}]$, $k = 1\,\text{N/m}$, and $M = 0.5\,\text{kg}$. What happens if we apply a constant force $u(t) = 1\,[\text{N}]$ at $t = 0$, assuming that the block is initially at rest?

$$U(s) = \mathcal{L}\{u(t)\} = \mathcal{L}\{1\} = \frac{1}{s},$$
$$Y(s) = G(s)U(s) \quad \Rightarrow \quad Y(s) = \left[\frac{2}{s^2+s+2}\right]\left(\frac{1}{s}\right).$$

The denominator polynomial of the resulting rational function is $s(s^2+s+2) = s[s-(-0.5+i\sqrt{1.75})][s-(-0.5-i\sqrt{1.75})] = s\left[(s+0.5)^2+1.75\right]$.
Using this information we can expand $Y(s)$ as a sum of partial fractions:

$$Y(s) = \frac{1}{s} - \frac{s+1}{s^2+s+2} = \frac{1}{s} - \frac{s+1}{(s+0.5)^2+1.75},$$
$$= \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+1.75} - \frac{0.5}{(s+0.5)^2+1.75},$$

Transfer Functions – Motivation VIII

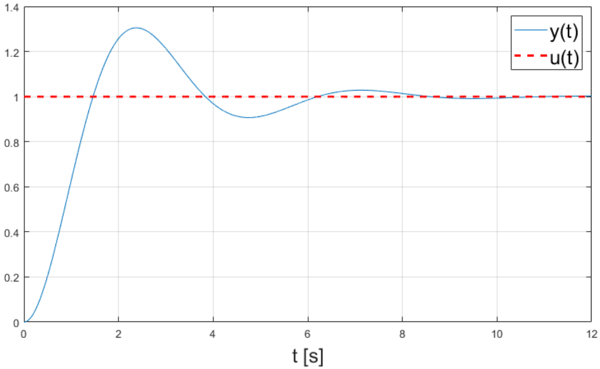
- By using the previous properties of the Laplace Transform (particularly the *Linearity* and *Frequency Shifting* Properties), and a Table of Laplace Transforms, we have that

$$Y(s) = \underbrace{\frac{1}{s}}_{1} - \underbrace{\frac{s+0.5}{(s+0.5)^2+1.75}}_{e^{-0.5t}\cos(\sqrt{1.75}t)} - \underbrace{\frac{0.5}{\sqrt{1.75}}\left[\frac{\sqrt{1.75}}{(s+0.5)^2+1.75}\right]}_{\frac{0.5}{\sqrt{1.75}}\left[e^{-0.5t}\sin(\sqrt{1.75}t)\right]},$$

Transfer Functions – Motivation IX

And we have a typical response from an underdamped second order linear system:

$$y(t) = 1 - e^{-0.5t}\cos(\sqrt{1.75}t) - \frac{0.5}{\sqrt{1.75}}e^{-0.5t}\sin(\sqrt{1.75}t).$$



Transfer Functions Definition

Definition: Transfer Function

The $p \times m$ matrix $G(s)$ of rational functions of the variable s that relates the Laplace Transform of the input $U(s) \in \mathbb{C}^m$ to the Laplace Transform of the output $Y(s) \in \mathbb{C}^p$, for a LTI system with zero initial conditions, such that:

$$Y(s) = G(s)U(s)$$

- There is another interpretation to the Transfer Function (the Laplace Transform of the system's impulse response), but we are not going to use it.
- We will see later that there is also a very interesting interpretation of the Transfer Function as a "generalized frequency-dependent gain".

Transfer Functions in MATLAB I

- It is quite easy to represent LTI systems by their Transfer Functions in MATLAB:

```
% Transfer Functions can be represented
% using the coefficients of the numerator and denominator
% polynomials.
%
% For example: n(s) = 2, d(s) = s^2 + s + 2.

G = tf([2],[1 1 2])
```

Output:

```

G =

        2
  -----------
  s^2 + s + 2

Continuous-time transfer function.
```

Transfer Functions in MATLAB II

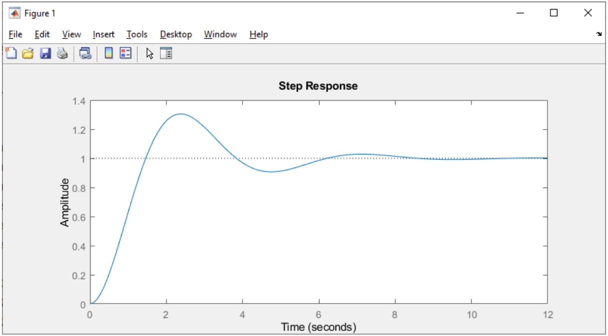
- This creates an object from the class `tf` (Transfer Function) with the attributes:

```
get(G)
        Numerator: {[0 0 2]}
      Denominator: {[1 1 2]}
         Variable: 's'
          IODelay: 0
       InputDelay: 0
      OutputDelay: 0
               Ts: 0
         TimeUnit: 'seconds'
        InputName: {''}
        InputUnit: {''}
       InputGroup: [1x1 struct]
       OutputName: {''}
       OutputUnit: {''}
      OutputGroup: [1x1 struct]
            Notes: [0x1 string]
         UserData: []
             Name: ''
     SamplingGrid: [1x1 struct]
```

Transfer Functions in MATLAB III

- This object allows for direct manipulation and simulation of common situations such as the application of unit step inputs (the same scenario considered previously in the mass-spring-damper example):

```
step(G)
```



Transfer Functions: Frequency Response I

- A Transfer Function $G(s)$ can also be seen as a "generalized gain" that varies with the frequency s.
- Indeed, we can determine the **long-term behavior** of the system output as a result of being driven by sinusoidal inputs with frequency ω^* just by examining the absolute value and the argument of the complex number $G(i\omega^*) = G(s)\big|_{s=i\omega^*}$.

Stable LTI systems excited by sinusoidal signals with frequency ω^* will have as outputs, after a sufficiently long time, sinusoidal signals with the same frequency ω^*, and:

1. Output amplitude: $|Y(i\omega^*)| = \underbrace{|G(i\omega^*)|}_{\text{Amplitude gain}} \underbrace{A_{\text{in}}}_{\text{Input amplitude}}$,
2. Output phase: $\arg\{Y(i\omega^*)\} = \underbrace{\arg\{G(i\omega^*)\}}_{\text{Additional phase}} + \underbrace{\phi_{\text{in}}}_{\text{Input phase}}$,

Transfer Functions: Frequency Response II

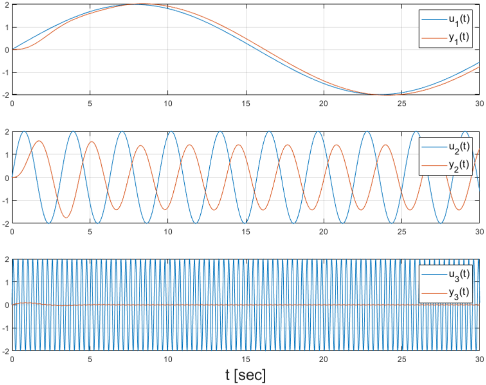
- Consider the following simulations in MATLAB in which the amplitude of the input is kept constant, but the frequency is changed:

```
% Frequency response example.
clc;
close all;

G = tf(2,[1 1 2]);

t = linspace(0,30,10e3);

u1 = 2*sin(0.2*t);
u2 = 2*sin(2*t);
u3 = 2*sin(20*t);

y1 = lsim(G,u1,t);
y2 = lsim(G,u2,t);
y3 = lsim(G,u3,t);
```

Transfer Functions: Frequency Response III

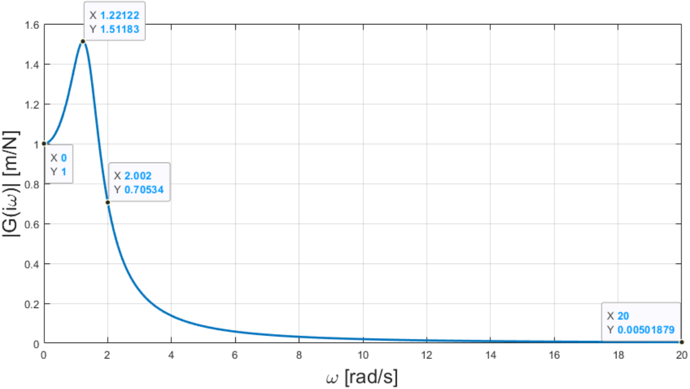
We can predict the amplitude of the output signal after the transient period:

$$G(i0.2) = 1.0099 - i0.1031 \Rightarrow \text{Gain:} \quad |G(i0.2)| = 1.015.$$
$$G(i2) = -0.5 - i0.5 \Rightarrow \text{Gain:} \quad |G(i2)| = 0.707.$$
$$G(i20) = 0.005 - i0.0003 \Rightarrow \text{Gain:} \quad |G(i20)| = 0.005.$$



Transfer Functions: Frequency Response IV

Actually, it is easy to compute the gains of the system for each excitation frequency:



Notice that there is a resonance around $\omega = 1.22\,\mathrm{rad/s}$.

Transfer Functions: Frequency Response V

- From this last property, we can determine what is the long-term behavior of the output when we have a constant input driving a stable LTI system: you have just to imagine that this constant signal is a sinusoidal signal with frequency $\omega^* = 0$, such as $u(t) = a = a\sin(0t + \pi/2)$:

$$\text{Long-term output: } |Y(0)| = G(0)\, a,$$

  and $G(0)$ is called the **DC-gain** of the system (DC = Direct Current, an Electric Engineering term used for gain at zero frequency).
- In the last example ($G = \frac{2}{s^2+s+2}$) we know that $G(0) = 1$ and the system is stable. Therefore the final value the output will reach, after the initial transient period, has the same numerical value of the constant input value. Check the previous figures on the step response to verify this claim!

Transfer Functions: General Case I

- The Transfer Function $G(s)$ can be directly obtained from any state-space representation of the LTI system, in the following way:

$$\begin{aligned}
\mathcal{L}\{\dot{x}\} &= \mathcal{L}\{Ax + Bu\},\\
s\mathcal{L}\{x\} - \cancelto{0}{x(0)} &= A\mathcal{L}\{x\} + B\mathcal{L}\{u\},\\
sX(s) &= AX(s) + BU(s),\\
[s\mathbb{1} - A]X(s) &= BU(s),\\
X(s) &= [s\mathbb{1} - A]^{-1}BU(s),
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{y\} &= \mathcal{L}\{Cx + Du\},\\
\mathcal{L}\{y\} &= C\mathcal{L}\{x\} + D\mathcal{L}\{u\},\\
Y(s) &= CX(s) + DU(s),
\end{aligned}$$

Therefore, $Y(s) = \left\{C[s\mathbb{1} - A]^{-1}B + D\right\}U(s)$, and

$$\boxed{G(s) = C[s\mathbb{1} - A]^{-1}B + D.}$$

Transfer Functions: General Case II

- It is interesting to notice that

$$[s\mathbb{1} - A]^{-1} = \frac{1}{\det\{s\mathbb{1} - A\}} \mathrm{Adj}\{s\mathbb{1} - A\},$$

such that

$$G(s) = C\frac{[\mathrm{Adj}\{s\mathbb{1} - A\}]}{\det\{s\mathbb{1} - A\}}B + D$$

where $\mathrm{Adj}\{M\}$ is the so-called Adjugate Matrix of M, which is the transpose of the Cofactor Matrix of M: a $n \times n$ matrix whose elements are the determinants, multiplied by $(-1)^{i+j}$, of $(n-1) \times (n-1)$ submatrices left after removing the i-th row and j-th column of the matrix M.

Transfer Functions: General Case III

- As a result, we can show that $\mathrm{Adj}\{s\mathbb{1} - A\}$ will be a matrix of polynomial functions of order at most $n-1$. And $\det\{s\mathbb{1} - A\}$ will be a polynomial of order n.
- Therefore, the general expression for $G(s)$ is a $p \times m$ matrix of rational functions of the variable s, all of them with the same polynomial denominator (if we do not allow further simplifications):

$$G(s) = \begin{bmatrix} \frac{z_{11}(s)}{d(s)} & \frac{z_{12}(s)}{d(s)} & \cdots & \frac{z_{1m}(s)}{d(s)} \\ \frac{z_{21}(s)}{d(s)} & \frac{z_{22}(s)}{d(s)} & \cdots & \frac{z_{2m}(s)}{d(s)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{z_{p1}(s)}{d(s)} & \frac{z_{p2}(s)}{d(s)} & \cdots & \frac{z_{pm}(s)}{d(s)}, \end{bmatrix}, \quad d(s) = \det\{s\mathbb{1} - A\},$$

where p is the number of outputs, and m is the number of inputs.

Transfer Functions: General Case IV

- Notice that each element in the $G(s)$ matrix, such as

$$G_{ij}(s) = \frac{z_{ij}(s)}{d(s)},$$

is a "generalized gain" in the frequency-domain that relates the input $U_j(s)$ to the output $Y_i(s)$, with

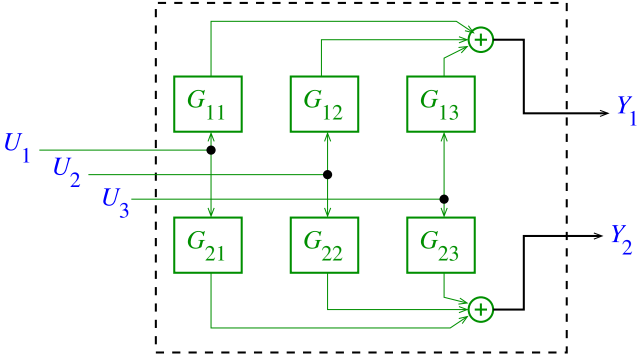
$$Y(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_p(s) \end{bmatrix}, \quad U(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_m(s) \end{bmatrix},$$

and

$$Y_i(s) = G_{i1}(s)U_1(s) + G_{i2}(s)U_2(s) + \cdots + G_{im}(s)U_m(s).$$

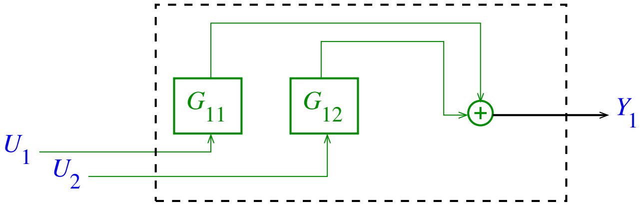
Transfer Functions: General Case V

Example of a system $G(s)$ with 3 inputs and 2 outputs:



Transfer Functions: General Case VI

Example for the Cessna 182 Longitudinal Model with 2 inputs (Elevator deflection and Thrust command) and 1 output (True Airspeed), trimmed at $V_{\mathrm{T}} = 100\,\mathrm{km/h} \approx 54\,\mathrm{knots}$ and $h = 1,140\,\mathrm{m} \approx 3740\,\mathrm{ft}$:



$$G_{11} = \frac{-0.1696s^3 + 82.53s^2 + 174.1s + 2.46\times10^{-5}}{s^5 + 4.222s^4 + 12.26s^3 + 0.6056s^2 + 1.119s - 2.187\times10^{-16}},$$
$$G_{12} = \frac{9.908s^4 + 41s^3 + 108.6s^2 - 24.63s + 3.971\times10^{-5}}{s^5 + 4.222s^4 + 12.26s^3 + 0.6056s^2 + 1.119s - 2.187\times10^{-16}}.$$

Transfer Functions: General Case VII

- Given a rational function $G_{ij}(s)$, we say that

$$G_{ij}(s) = \frac{n(s)}{d(s)} \quad \Rightarrow \begin{cases} \text{roots of } n(s) \rightarrow \text{Zeros of } G_{ij}(s) \\ \text{roots of } d(s) \rightarrow \text{Poles of } G_{ij}(s) \end{cases}$$

with $n(s)$ the numerator polynomial, and $d(s)$ the denominator polynomial. In general, we can write that

$$G_{ij}(s) = \frac{n(s)}{d(s)} = K\frac{(s-z_1)(s-z_2)\cdots(s-z_q)}{(s-p_1)(s-p_2)\cdots(s-p_n)},$$

with $K \in \mathbb{R}$ a constant, and $z_1, z_2, \ldots, z_q$ the zeros of $G_{ij}(s)$, and $p_1, p_2, \ldots, p_n$ the poles of $G_{ij}(s)$.

Warning: if a zero and a pole are equal (or very close in practice); i.e., $z_i \approx p_j$ for some i and j; then there can be a **Zero/Pole cancellation**.

Transfer Functions: General Case VIII

- From the previous discussion, without considering possible further simplifications, the poles of the Transfer Function must be values s that satisfy

$$d(s) = \det\{s\mathbb{1} - A\} = 0,$$

  and this means that the roots of $d(s)$ are the values of s that render the matrix $(s\mathbb{1} - A)$ singular (i.e., there is no associated inverse matrix).

- Notice that, by definition, the eigenvalues λ of matrix A associated with non-zero eigenvectors v are such that

$$Av = \lambda v \quad \Rightarrow \quad (\lambda\mathbb{1} - A)v = 0,$$

  and in order to have non-zero solutions for the eigenvectors v, one needs to guarantee that

$$d(\lambda) = \det\{\lambda\mathbb{1} - A\} = 0.$$

Transfer Functions: General Case IX

- Therefore, if there is no cancellations of Zeros and Poles, the poles of the Transfer Function are precisely the eigenvalues of the matrix A! Consequently, in this case, by looking at the poles of the Transfer Function, we can determine if the system is dynamically stable or not.
- In the general case, we have that

$$\text{Poles}\{G_{ij}(s)\} \subseteq \text{Eigenvalues}\{A\}$$

Transfer Functions: Residues and Model Order Reduction I

- The poles of the Transfer Function of a LTI system are as important as the eigenvalues of the matrix A.
- Besides helping to determine the dynamic stability of the Local LTI model of the aircraft, we can consider that each real pole and each pair of complex conjugate poles constitute a **dynamic mode of response**, in the same way we considered the eigenvalues of the matrix A.

Transfer Functions: Residues and Model Order Reduction II

- This can be better revealed by considering the Partial Fraction Decomposition of the Transfer Function $G_{ij}(s)$. When there are no repeated poles (or eigenvalues), the decomposition generates:

$$G_{ij}(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_q)}{(s-p_1)(s-p_2)\cdots(s-p_n)},$$

$$G_{ij}(s) = D_{ij} + \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \frac{r_3}{s-p_3} + \cdots + \frac{r_n}{s-p_n},$$

with $r_k \in \mathbb{C}$, $k = 1, 2, \ldots, n$, the so-called "residue" associated with the pole $p_k \in \mathbb{C}$.

Transfer Functions: Residues and Model Order Reduction III

- When the system is excited by a nonzero input $U_j(s)$, the output $Y_i(s)$ will be the sum of the contributions of each mode of response to the final response:

$$
\begin{aligned}
Y_i(s) &= G_{ij}(s)U_j(s), \\
\\
&= D_{ij}U_j(s) \\
&+ \left(\frac{r_1}{s-p_1}\right)U_j(s) \\
&+ \left(\frac{r_2}{s-p_2}\right)U_j(s) \\
&+ \cdots + \left(\frac{r_n}{s-p_n}\right)U_j(s),
\end{aligned}
$$

such that their relative importance can be assessed by comparing the values of the resiudes r_k.

Transfer Functions: Residues and Model Order Reduction IV

- For example, if $\mathrm{abs}\{r_1\} \ll \mathrm{abs}\{r_k\}$, for all $k \neq 1$, the contribution of the mode of response 1 could be eventually neglected.
- Notice that, since the residues associated with complex conjugate poles are also complex conjugate, the elimination of a complex pole due to its negligible residue will automatically lead to the elimination of the corresponding conjugate pole. That is, *we always eliminate real poles, or pairs of complex conjugate poles*, never just one complex pole.
- The possible elimination of a mode of response from the dynamics of the system reflects an “almost cancellation” of poles and zeros.

Transfer Functions: Residues and Model Order Reduction V

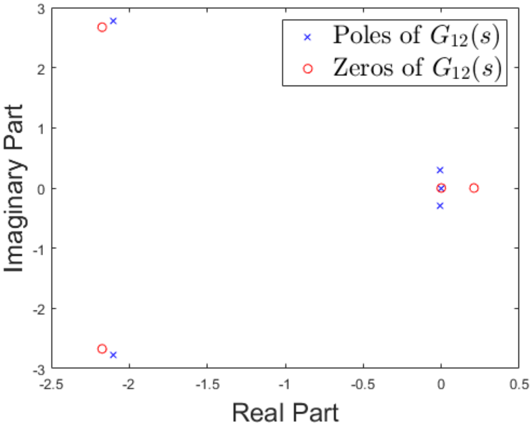
- Example: Cessna 182 Longitudinal Dynamics.
  - In MATLAB, we can obtain the **Zero-Pole-Gain representation** for a Transfer Function by using the command G = **zpk**(sys), with sys a **tf** (transfer function) or a **ss** (state space) object:

```
G =

From input "delta_elev" to output "VTAS":
    -0.16963 (s-488.6) (s+2.101) (s+1.413e-07)
--------------------------------------------------
s (s^2 + 0.0179s + 0.09258) (s^2 + 4.204s + 12.09)

From input "delta_thrust" to output "VTAS":
9.9078 (s-0.2095) (s-1.612e-06) (s^2 + 4.348s + 11.87)
--------------------------------------------------------
    s (s^2 + 0.0179s + 0.09258) (s^2 + 4.204s + 12.09)

Continuous-time zero/pole/gain model.
```

  - Notice that the pairs of complex conjugate poles and zeros were grouped together in second order polynomials for both rational functions G_{11} and G_{12}.

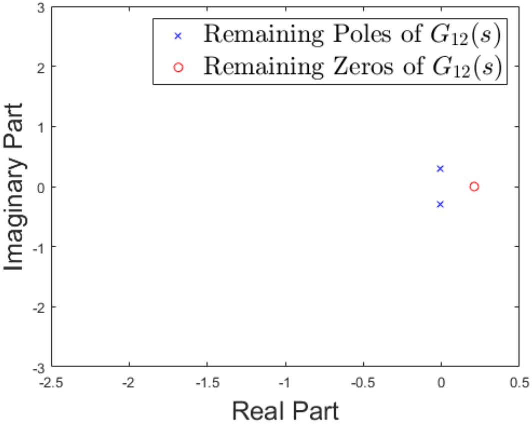
Transfer Functions: Residues and Model Order Reduction VI

- We can better see how close some zeros are from some poles, leading to "almost cancellations", by using the command **pzmap**(G), particularly for $G_{12}(s)$:



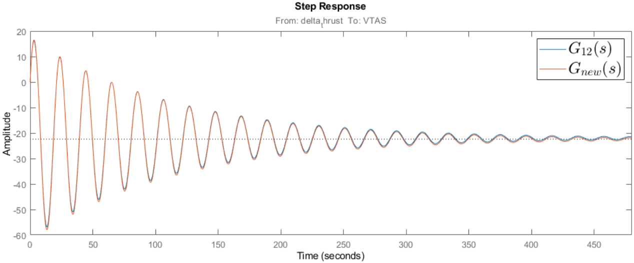
Transfer Functions: Residues and Model Order Reduction VII

- The process of eliminating modes can be automated in MATLAB by using the command Gnew = **minreal**(G,0.1), in which we selected the cancellation of zeros and poles within a distance of 0.1 from each other:



Transfer Functions: Residues and Model Order Reduction VIII

- A comparison of the step-responses (abrupt change in the Thrust command leading to changes in the True Airspeed) for the original model $G_{12}(s)$ and the new model is provided below:



$$G_{12} = \frac{9.9078(s - 0.2095)(s - 1.612 \times 10^{-6})(s^2 + 4.348s + 11.87)}{s(s^2 + 0.0179s + 0.09258)(s^2 + 4.204s + 12.09)},$$

$$G_{\text{new}} = \frac{9.9078(s - 0.2095)}{(s^2 + 0.0179s + 0.09258)}, \quad \rightarrow \textbf{Much Simpler!!}$$

Transfer Functions: Residues and Model Order Reduction IX

- We should be extra-careful when peforming such cancellations. We can accidentally cancel a pole on the right-hand side of the complex plane, that is, we can accidentally cancel an **unstable pole**!
- Even if the residue associated to an unstable mode of response is very small, the presence of the unstable mode would become evident sooner or later because the unstable mode grows exponentially in the time-domain.
- In the present example, we cancelled a pole at zero with a zero on the right-hand side very close to it ($z = 1.612 \times 10^{-6}$). No cancellations of right-hand side poles were effected.

  To cancel right-hand poles is forbidden!

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses I

- In the Partial Fractions Decomposition of $G_{ij}(s)$, very often we group every pair of fractions associated with complex conjugate poles in just one second order fraction, while taking into consideration that the corresponding residues are also complex conjugate numbers, such that

$$G_{ij}(s) = D_{ij} + \frac{r_1}{s - p_1} + \frac{\alpha_{23} s + \beta_{23}}{(s - \sigma_2)^2 + \omega_2^2} + \frac{r_4}{s - p_4} + \cdots$$

and all the coefficients in the numerator polynomials will be real numbers. In this example it was assumed that $p_2 = \mathrm{conj}\{p_3\}$, and therefore $\alpha_{23} = 2\mathrm{Re}\{r_2\} = 2\mathrm{Re}\{r_3\}$, and $\beta_{23} = -2\mathrm{Re}\{r_2 p_3\}$.

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses II

- This means that we can have a good understanding about how the output $Y_i(s)$ of the LTI system will respond to abrupt changes (step changes) in the input $U_j(s)$ by knowing how first-order and second-order systems respond to this kind of input $U_j(s) = \frac{1}{s}$ (unit step input):

$$
\begin{aligned}
Y_i(s) = D_{ij}\frac{1}{s} \\
&+ \left[\left(\frac{r_1}{s-p_1}\right)\frac{1}{s}\right] \\
&+ \left[\left(\frac{\alpha_{23}s+\beta_{23}}{(s-\sigma_2)^2+\omega_2^2}\right)\frac{1}{s}\right] \\
&+ \left[\left(\frac{r_4}{s-p_4}\right)\frac{1}{s}\right] + \cdots
\end{aligned}
$$

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses III

- For 1^{st} Order modes of response, we have that

$$Y(s) = \left(\frac{r}{s-p}\right)\frac{1}{s} = \left(\frac{G_{\mathrm{dc}}}{\tau s + 1}\right)\frac{1}{s},$$

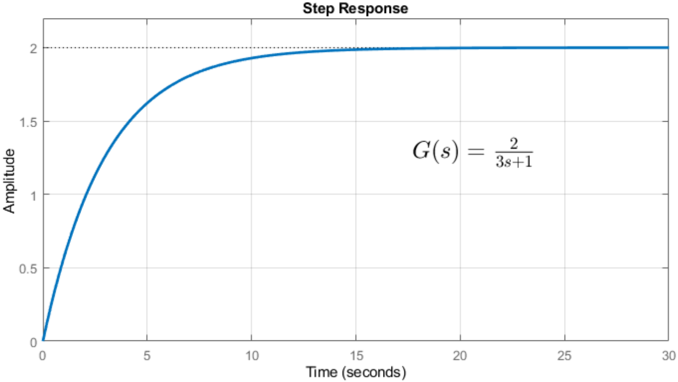
  with the **DC-Gain** $G_{\mathrm{dc}} = -r/p$, and the **time constant** $\tau = -1/p$.

- The step-response is given by

$$y(t) = G_{\mathrm{dc}}\left(1 - e^{-t/\tau}\right),$$

  and usually one assumes that after $t_{\mathrm{s}} = 5\tau$ the output will be in steady-state (notice that, mathematically, the output never completely settles down).

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses IV



Notice that the final variation of the output (its amplitude) is equal to G_{dc} multiplied by the input's amplitude, and the Settling Time $t_{\mathrm{s}} = 5\tau$.

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses V

- For 2^{nd} Order dynamic modes one usually considers a simplified behavior (called a standard second-order system) that captures much of the essential information:

$$Y(s) = \left(\frac{G_{\mathrm{dc}} \omega_{\mathrm{n}}^2}{s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2} \right) \frac{1}{s},$$

where

$$(s-\sigma)^2 + \omega^2 = s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2,$$

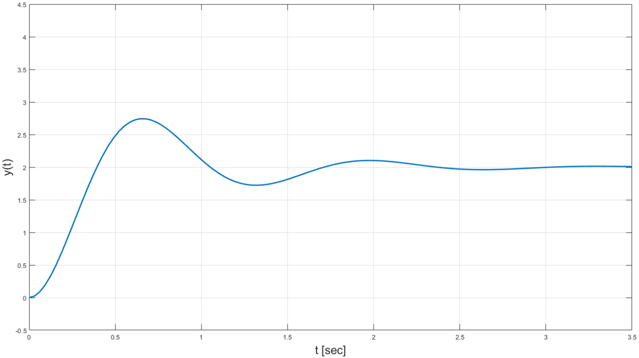
and, therefore,

$$\text{Real Part: } \sigma = -\zeta\omega_{\mathrm{n}},$$

$$\text{Imaginary Part: } \omega = \omega_{\mathrm{n}}\sqrt{1-\zeta^2}.$$

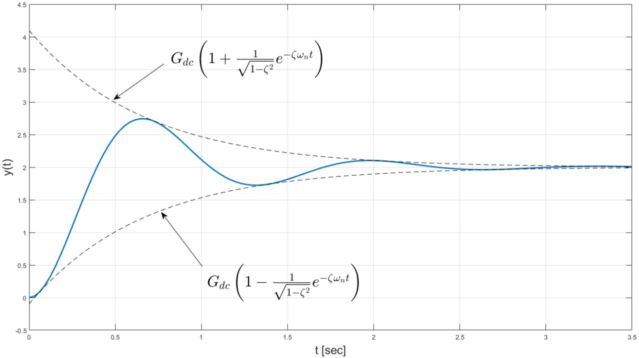
with the dimensioless **Damping Ratio** ζ, and the **Natural Frequency** ω_{n} in [rad/s]. Together with the **DC-Gain** G_{dc}, we can describe the step-response relying only on these 3 parameters.

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses VI



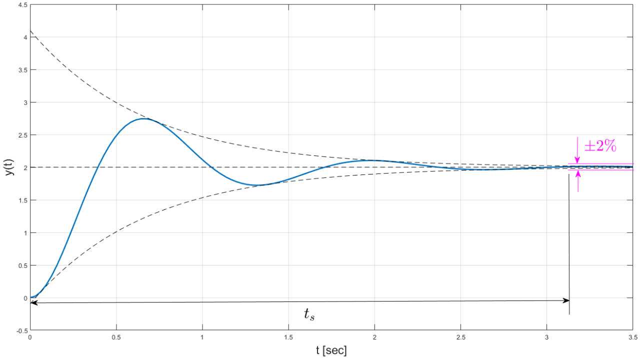
$$y(t) = G_{\text{dc}}\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_{\text{n}} t}\sin(\omega_{\text{n}}\sqrt{1-\zeta^2}\, t + \phi)\right); \phi = \cos^{-1}(\zeta).$$

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses VII



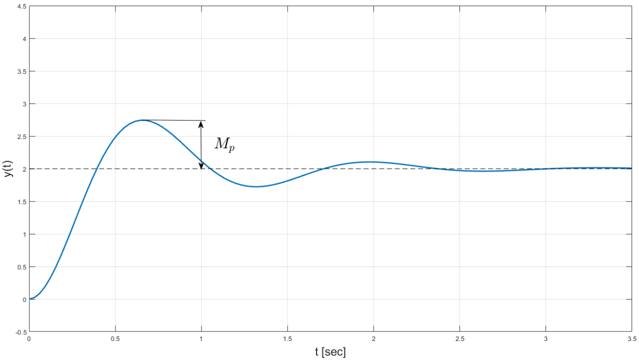
$$y(t) = G_{\text{dc}}\left(1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_{\text{n}}t}\sin(\omega_{\text{n}}\sqrt{1-\zeta^2}\,t + \phi)\right); \phi = \cos^{-1}(\zeta).$$

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses VIII



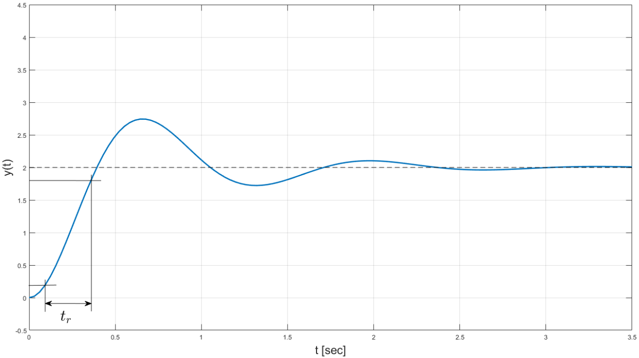
Settling time (for 2%): $t_{\mathrm{s}} \approx \dfrac{4}{\zeta\omega_n}$

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses IX



$$\text{Percent Overshoot: } M_{\text{p}} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \approx 1 - \frac{\zeta}{0.6}$$

Transfer Functions: 1^{st} and 2^{nd} Order Step Responses X



$$\text{Rise time (from 10\% to 90\%): } t_{\text{r}} \approx \frac{1.8}{\omega_n}$$

Aircraft Dynamics Requirements I

- From what we discussed about 1^{st} and 2^{nd} order systems step-responses, we can say that:
  1. Dynamic Modes described by real poles:
     - If the Time Constant τ can be reduced, the speed of the response can be improved.
  2. Dynamic Modes described by complex conjugate pairs:
     - As far left are the poles from the imaginary axis in the complex plane, as faster the system will respond ($\zeta\omega_{\mathrm{n}} \rightarrow \infty \Rightarrow t_{\mathrm{s}} \rightarrow 0$).
     - If the daming ratio $0 < \zeta \leq 1$ increases, the Percent Overshoot will decrease (there will be more damping).
     - The Natural Frequency determines the Rise Time. Greater values for ω_{n} will decrease t_{r}.

Aircraft Dynamics Requirements II

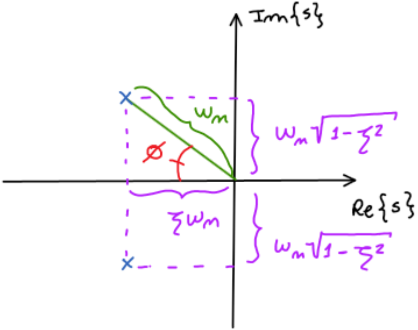
- Therefore, if we need to design the aircraft such that requirements on
  1. Maximum Settling Time $t_{\mathrm{s}}^{\max}$,
  2. Maximum Percent Overshoot $M_{\mathrm{p}}^{\max}$, and
  3. Maximum Rise Time $t_{\mathrm{r}}^{\max}$

  are specified, we can use the former relations to find where the poles (or eigenvalues) of the system should be located.

Aircraft Dynamics: Desired Poles Locations

- 2^{nd} Order modes of response can be characterized by complex conjugate poles (or eigenvalues) that can be expressed as

$$p_{1,2} = -\zeta\omega_{\text{n}} \pm i\omega_{\text{n}}\sqrt{1-\zeta^2}$$



Notice that:

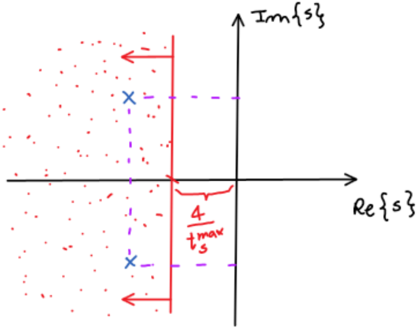
$$(\zeta\omega_{\text{n}})^2 + \left(\omega_{\text{n}}\sqrt{1-\zeta^2}\right)^2 = \omega_{\text{n}}^2$$

$$\zeta = \cos(\phi)$$

Aircraft Dynamics: Desired Poles Locations

- 2^{nd} Order modes of response can be characterized by complex conjugate poles (or eigenvalues) that can be expressed as

$$p_{1,2} = -\zeta\omega_{\text{n}} \pm i\omega_{\text{n}}\sqrt{1-\zeta^2}$$



Notice that:

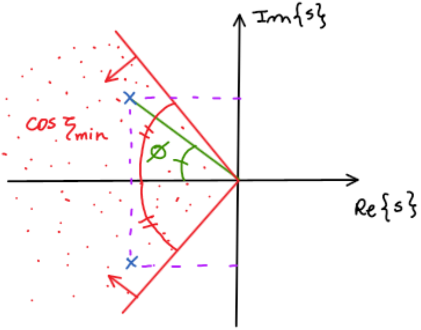
$$t_{\text{s}} \approx \frac{4}{\zeta\omega_{\text{n}}} \leq t_{\text{s}}^{\max}$$

$$\zeta\omega_{\text{n}} \geq \frac{4}{t_{\text{s}}^{\max}}$$

Aircraft Dynamics: Desired Poles Locations

- 2^{nd} Order modes of response can be characterized by complex conjugate poles (or eigenvalues) that can be expressed as

$$p_{1,2} = -\zeta\omega_{\rm n} \pm i\omega_{\rm n}\sqrt{1-\zeta^2}$$



Notice that:

$$M_{\rm p} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \approx 1 - \frac{\zeta}{0.6}$$

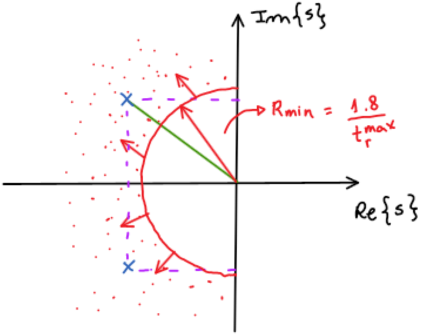
$$M_{\rm p} \leq M_{\rm p}^{\max}$$

$$\zeta \gtrsim \underbrace{0.6(1 - M_{\rm p}^{\max})}_{\zeta_{\min}}$$

Aircraft Dynamics: Desired Poles Locations

- 2^{nd} Order modes of response can be characterized by complex conjugate poles (or eigenvalues) that can be expressed as

$$p_{1,2} = -\zeta\omega_{\text{n}} \pm i\omega_{\text{n}}\sqrt{1-\zeta^2}$$



Notice that:

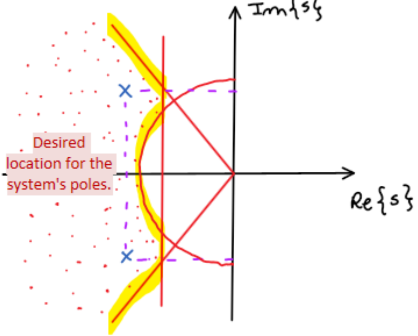
$$t_{\text{r}} \approx \frac{1.8}{\omega_{\text{n}}}$$
$$t_{\text{r}} \leq t_{\text{r}}^{\max}$$

$$\omega_{\text{n}} \gtrsim \underbrace{\frac{1.8}{t_{\text{r}}^{\max}}}_{R_{\min}}$$

Aircraft Dynamics: Desired Poles Locations

- 2^{nd} Order modes of response can be characterized by complex conjugate poles (or eigenvalues) that can be expressed as

$$p_{1,2} = -\zeta\omega_{\text{n}} \pm i\omega_{\text{n}}\sqrt{1-\zeta^2}$$



Notice that:

$$\zeta\omega_{\text{n}} \geq \frac{4}{t_{\text{s}}^{\max}}$$

$$\zeta \gtrsim \underbrace{0.6(1 - M_{\text{p}}^{\max})}_{\zeta_{\min}}$$

$$\omega_{\text{n}} \gtrsim \underbrace{\frac{1.8}{t_{\text{r}}^{\max}}}_{R_{\min}}$$

Section 4

Laplace Transform Interpretations

Subsection 1

Laplace Transform as a “Generalized Fourier Transform”

Fourier Series I

- Every periodic signal in the time-domain, with period T; i.e., $u(t) = u(t+T)$; can be represented as an infinite sum of sines and cosines:

$$u(t) = \sum_{n=0}^{\infty} A(n)\cos\left(n\frac{2\pi}{T}t\right) + B(n)\sin\left(n\frac{2\pi}{T}t\right),$$

with $A(n), B(n) \in \mathbb{R}$.

Fourier Series II

- By recognizing that $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$, and defining $\omega_0 = \frac{2\pi}{T}$, we can rewrite the previous expression as

$$\boxed{u(t) = \sum_{n=-\infty}^{\infty} C(n) e^{i(n\omega_0)t}}$$

where $C(n) = \frac{A(n) - iB(n)}{2}$ for $n \geq 0$, and $C(n) = \frac{A(n) + iB(n)}{2}$ for $n < 0$. Notice that $C(n)$ can be a complex number for each n (if $B(n) \neq 0$), i.e., $C(n) \in \mathbb{C}$.

Fourier Series III

- To compute the complex coefficients $C(n)$, we can multiply the previous expression by $e^{-i(n\omega_0)t}$ on both sides and integrate, such that

$$u(t)e^{-i(n\omega_0)t} = \sum_{m=-\infty}^{\infty} C(m)e^{i(m\omega_0)t}e^{-i(n\omega_0)t},$$

$$\int_0^T u(t)e^{-i(n\omega_0)t}dt = \sum_{m=-\infty}^{\infty} \int_0^T C(m)e^{i(m-n)\omega_0 t}dt$$

and only when $m = n$ there will be a non-oscillatory component whose integral is $C(n)T = C(n)\frac{2\pi}{\omega_0}$. For all other terms the corrresponding integral is zero, and therefore

$$C(n) = \frac{\omega_0}{2\pi} \int_0^T u(t)e^{-i(n\omega_0)t}dt$$

Fourier Series IV

- In preparation for what we are going to do next, we can define the coefficients $U(n\omega_0) = C(n)\frac{2\pi}{\omega_0} = C(n)T$, which are functions of the product $n\omega_0$, such that the pair of expressions to transform from one domain to another become

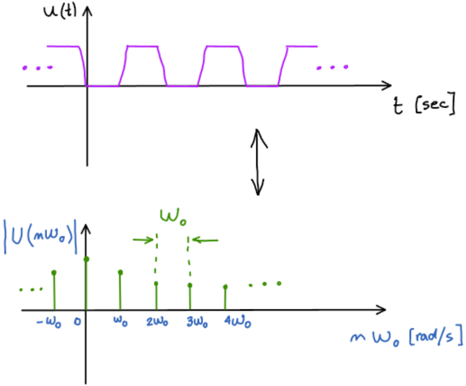
$$u(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} U(n\omega_0)\omega_0 e^{i(n\omega_0)t}$$

$$U(n\omega_0) = \int_0^T u(t) e^{-i(n\omega_0)t} dt$$

- From the last two expressions, we see that a **periodic** signal in the time-domain t (in [sec]) is equivalent to a countable sequence of coefficients in the frequency-domain $n\omega_0$ (in [rad/s]):

$$u(t) \quad \longleftrightarrow \quad U(n\omega_0)$$

Fourier Series V



See this video (3Blue1Brown) for a magnificent application (and explanation) of the Fourier Series!

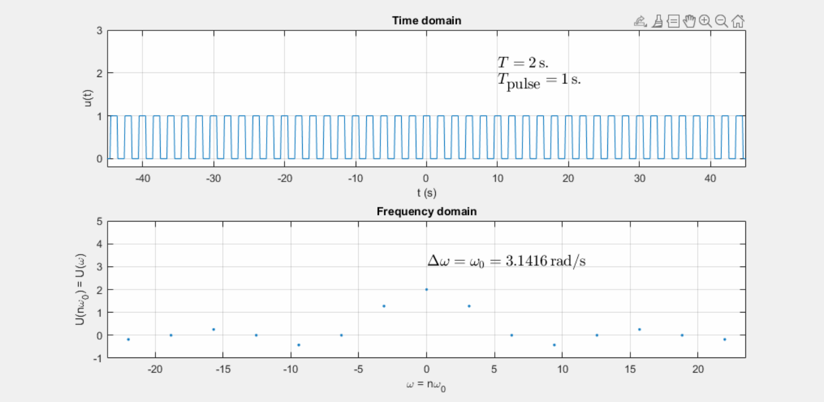
From the Fourier Series to the Fourier Transform I

- What if $u(t)$ is not a periodic signal? That is, what if $u(t) \neq u(t+T)$ for some $T > 0$?
- In this case, we can use the previous expressions and recognize that a non-periodic signal could be approximated by a periodic one whose period is very large, that is

$$T \to \infty.$$

From the Fourier Series to the Fourier Transform

Below you can see what happens with the Fourier Series coefficients as the period T increases for a periodic train of pulses, each pulse with duration T_{pulse}:



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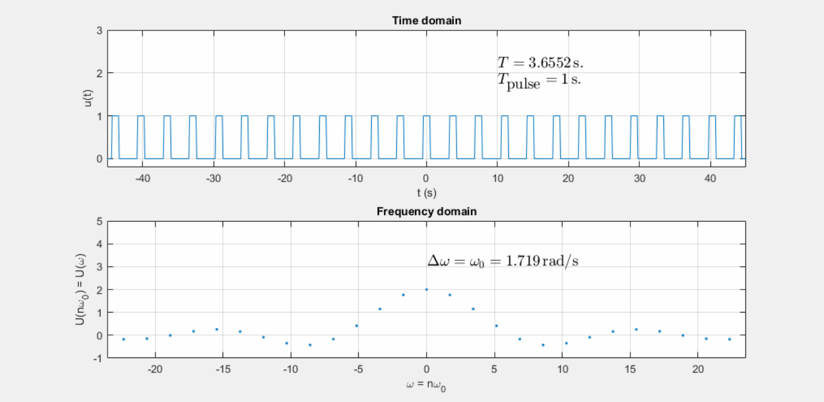

From the Fourier Series to the Fourier Transform

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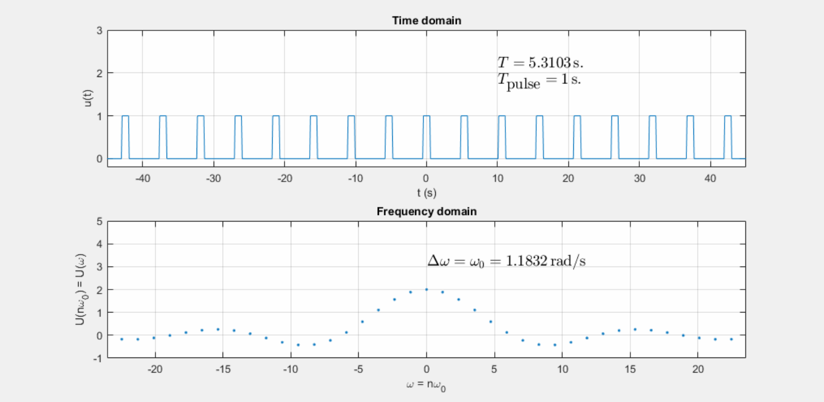

From the Fourier Series to the Fourier Transform

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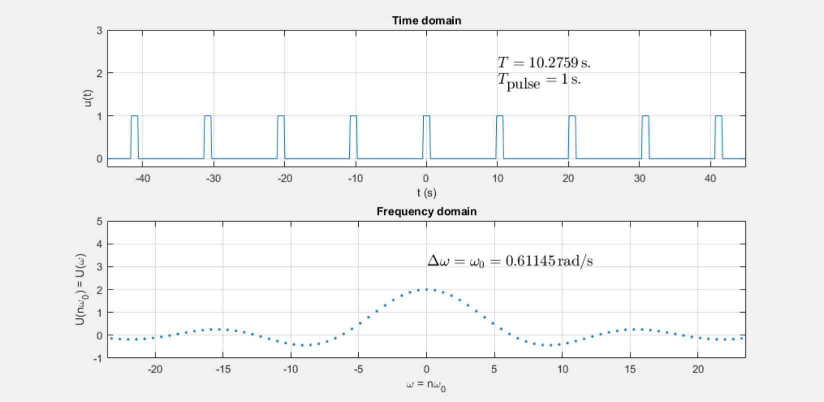
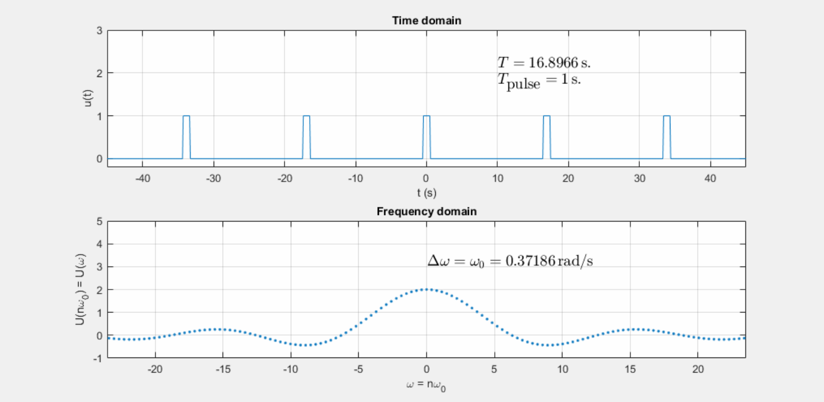


From the Fourier Series to the Fourier Transform

Below you can see what happens with the Fourier Series coefficients as the period T increases for a periodic train of pulses, each pulse with duration T_{pulse}:



From the Fourier Series to the Fourier Transform

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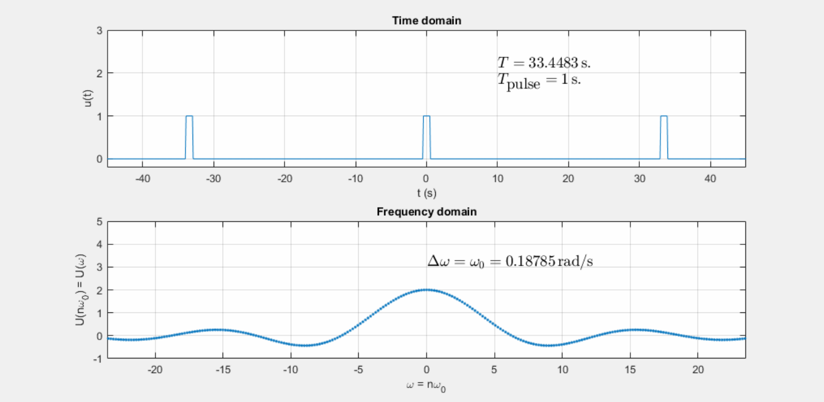
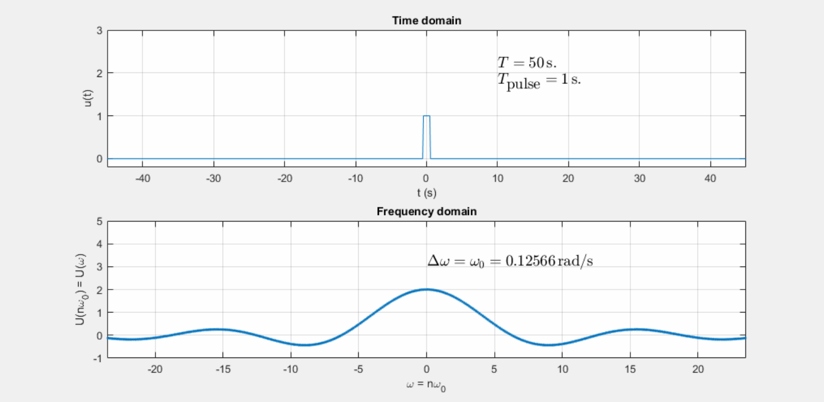


From the Fourier Series to the Fourier Transform

Below you can see what happens with the Fourier Series coefficients as the period T increases for a periodic train of pulses, each pulse with duration T_{pulse}:



Fourier Transform as the Limit of the Fourier Series I

- In this case, while it continues to be true that $n\omega_0 = \omega$, the separation $\Delta\omega = \omega_0$ between successive coefficients in the frequency-domain becomes progressively smaller, or "infinitesimal" in the limit:

$$\Delta\omega = \omega_0 \mapsto d\omega,$$

and the coefficients become *uncountable*. The summation in the Fourier Series expansion must then be substituted by an integration:

$$u(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} U(n\omega_0)e^{i(n\omega_0)t}\omega_0 \quad \rightarrow \quad u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)e^{i\omega t}d\omega.$$

Fourier Transform as the Limit of the Fourier Series II

- This leads to the **Fourier Transform Pair**:

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)e^{i\omega t}d\omega = \mathcal{F}^{-1}\left\{U(\omega)\right\}$$

$$U(\omega) = \int_{0}^{\infty} u(t)e^{-i\omega t}dt = \mathcal{F}\left\{u(t)\right\}$$

- A value $U(\omega^*)$ can be interpreted as the coefficient associated with the specific frequency $\omega = \omega^*$. The absolute value $|U(\omega^*)|$ is then proportional to the amplitude of the specific oscillation with frequency ω^* that composes the signal $u(t)$ in the time-domain. The angle or argument $\arg\left\{U(\omega^*)\right\}$ is the phase displacement of this oscillatory component.
  The continuum of values $U(\omega)$ is called the **Spectrum of the signal**.

Laplace Transform I

- Notice that the Fourier Transform is very interesting, but somewhat limited in the sense that the set of functions $u(t)$ that have well defined Fourier Transforms $U(\omega)$ is limited.
- For example,

$$u(t) = 1, \forall t \geq 0 \quad \Rightarrow \quad U(\omega) = \int_0^\infty e^{-i\omega t} dt = ??$$

  does not converge, because $e^{-i\omega t} = \cos(\omega t) + i \sin(\omega t)$ is an oscillatory term that does not go to zero.
- Actually, we can prove that a *sufficient* condition for the existence of a Fourier Transform is that $u(t)$ is absolutely integrable:

$$\int_0^\infty |u(t)| dt < \infty$$

Laplace Transform II

- We could achieve the absolute integrability condition if we worked with signals that go faster to zero.
- One way of having this situation is by using the following "exponential operator" $E_\sigma[u(t)]$ on time-domain signals $u(t)$, defined by the multiplication of the signal by an exponential function, such as:

$$\begin{aligned} \hat{u}(t) &= E_\sigma[u(t)], \\ &= u(t)e^{-\sigma t}, \\ u(t) &= E_\sigma^{-1}[\hat{u}(t)], \\ &= \hat{u}(t)e^{\sigma t} = E_{-\sigma}[\hat{u}(t)]. \end{aligned}$$

  with $\sigma \in \mathbb{R}$.
- The multiplication by an exponential term that goes to zero enables the fulfillment of the absolute integrability requirement.
- Notice that an "operator" sends functions on a certain domain to functions on the <u>same</u> domain (time-domain in this case).

Laplace Transform III

- Now we can define a new transform: the Fourier Transform of the "exponentially attenuated signal":

$$\mathcal{F}\left\{E_\sigma\left[u(t)\right]\right\} = \mathcal{F}\left\{\hat{u}(t)\right\} = \int_0^\infty \left(u(t)e^{-\sigma t}\right) e^{-i\omega t} dt$$

- In this case, notice that by the appropriate selection of $\sigma \in \mathbb{R}$, we can guarantee the convergence of the following integral

$$\hat{U}(\omega) = \int_0^\infty \hat{u}(t) e^{-i\omega t} dt = \int_0^\infty u(t) e^{-(\sigma + i\omega)t} dt$$

  for a broad class of functions, even for functions that grow exponentially to infinity such as $u(t) = e^{2t}$ (just select, for example, $\sigma = 3$).

Laplace Transform IV

- To recover the original signal $u(t)$, we just have to reverse the steps taken before:

$$\textbf{(i)} \quad \hat{u}(t) = \mathcal{F}^{-1}\left\{\hat{U}(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{U}(\omega)e^{i\omega t}d\omega,$$

$$\textbf{(ii)} \quad u(t) = E_{\sigma}^{-1}\left[\hat{u}(t)\right] = \hat{u}(t)e^{\sigma t} = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{U}(\omega)e^{i\omega t}d\omega\right]e^{\sigma t},$$

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{U}(\omega)e^{(\sigma+i\omega)t}d\omega,$$

i.e., **(i)** compute the inverse Fourier Transform, and **(ii)** apply the inverse exponential operator.

Laplace Transform V

- Combining the two previous transformations (from time-domain to frequency-domain and vice-versa), we can define the so-called *complex frequency variable*

$$s = \sigma + i\omega \quad \Rightarrow \quad ds = i d\omega,$$

and re-label $\hat{U}(\omega) = U(s)$. We then finally have the **Laplace Transform Pair**:

$$u(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} U(s) e^{st} ds = \mathcal{L}^{-1}\left\{U(s)\right\}$$

$$U(s) = \int_{0}^{\infty} u(t) e^{-st} dt = \mathcal{L}\left\{u(t)\right\}$$

where the constant c is chosen appropriately to guarantee that $\sigma = c$ is sufficient for the convergence of both integrals.

Laplace Transform VI

- This intrepretation is a powerful one, because it shows that if we select $s = i\omega$; i.e., if $\sigma = 0$; the Laplace Transform $U(s)$ becomes the Fourier Transform $U(i\omega)$.

Subsection 2

Laplace Transform as a “Generalized Power Series”

Laplace Transforms as Generalized Power Series I

Disclaimer: The following is based on the Lecture 19, from a MIT course on Differential Equations, presented in March, 31, 2003, by Prof. Arthur Mattuck, and available on YouTube.

Laplace Transforms as Generalized Power Series II

Consider the following functions and their **Power Series Expansions**:

- Exponential function:

$$F_1: \ \ e^{x} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{k!}x^k + \cdots$$

- Sine function:

$$F_2: \ \ \sin(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + (-1)^{2k+1}\frac{1}{(2k+1)!}x^{2k+1} + \cdots$$

- A polynomial function:

$$F_3: \ \ 2 + 3x - 4x^2 = 2 + 3x - 4x^2 + 0x^3 + 0x^4 + \cdots + 0x^k + \cdots$$

Laplace Transforms as Generalized Power Series III

We could say that the previous mathematical objects F_1, F_2 and F_3 can be represented in two different ways: as functions of a variable x, or as coefficientes that are themselves functions of a variable k:

- Exponential function:

$$F_1 : \quad x \mapsto e^x \qquad \longleftrightarrow \qquad k \mapsto a(k) = \frac{1}{k!}$$

- Sine function:

$$F_2 : \quad x \mapsto \sin(x) \qquad \longleftrightarrow \qquad k \mapsto a(k) = (-1)^{2k+1} \frac{1}{(2k+1)!}$$

- A polynomial function:

$$F_3 : \quad x \mapsto 2 + 3x - 4x^2 \qquad \longleftrightarrow \qquad k \mapsto a(k),$$

where $a(0) = 2$, $a(1) = 3$, $a(2) = -4$, and $a(k) = 0, \forall k \geq 3$.

Laplace Transforms as Generalized Power Series IV

- Consider the "set of real analytical functions of a real variable". One element of this set is a function:

$$\hat{f}(x) : \mathbb{R} \to \mathbb{R}$$

- We can define an "Equivalence Relation" between functions $\hat{f}(x)$ from this set and functions $a(k)$ (which are indeed sequences of real numbers):

$$\hat{f}(x) : \mathbb{R} \to \mathbb{R} \qquad \longleftrightarrow \qquad a(k) : \{0, 1, 2, \cdots\} \to \mathbb{R}$$

by using the following equation:

$$\boxed{\hat{f}(x) = \sum_{k=0}^{\infty} a(k) x^k}$$

Therefore, every time the above equation is true, we will say that $\hat{f}(x)$ is equivalent to $a(k)$, and we can denote this fact by $\hat{f}(x) \equiv a(k)$.

Laplace Transforms as Generalized Power Series V

- Now suppose we want to generalize this idea by creating an association between functions $a_{\mathrm{c}}(t)$ of a *continuous* variable t (instead of $a(k)$) and a function $\hat{f}_{\mathrm{c}}(x)$ defined in another domain to which x belong.
- A natural way of doing that is to consider substituting the summation by the integral:

$$\hat{f}(x) = \sum_{k=0}^{\infty} a(k) x^k \quad \rightsquigarrow \quad \hat{f}_{\mathrm{c}}(x) = \int_0^{\infty} a_{\mathrm{c}}(t) x^t dt$$

- A reasonable requirement at this point is that $|x| < 1$, since this would facilitate the convergence of the integral on the right-hand side by making $x^t \to 0$ as $t \to \infty$.

Laplace Transforms as Generalized Power Series VI

- Another natural development is to represent $x^t = \left[e^{\ln(x)}\right]^t = e^{t\ln(x)}$ (in terms of the exponential function) to facilitate mathematical manipulations:

$$\hat{f}_{\text{c}}(x) = \int_0^\infty a_{\text{c}}(t) e^{t\ln(x)} dt.$$

- Relying on the previous requirement that $|x| < 1$, this means that $\ln(|x|) < 0$, and we can define a new variable

$$s = -\ln(x), \quad |x| < 1,$$

such that for real and positive values of $0 < x < 1$, s will be a real and positive number. However, for $-1 < x < 0$, s will be a complex number. Let's allow x to be a complex number such that $|x| < 1$, and then s will assume whatever value we want in the complex domain

$$s = \sigma + i\omega, \quad \sigma, \omega \in (-\infty, +\infty) \quad \Leftrightarrow \quad s \in \mathbb{C}.$$

Laplace Transforms as Generalized Power Series VII

- With the previous definition we have that

$$\hat{f}_{\mathrm{c}}(x) = \int_0^{\infty} a_{\mathrm{c}}(t) e^{-st} dt = \hat{f}_{\mathrm{c}}\left(e^{-s}\right) = F(s).$$

- Substituting now $a_{\mathrm{c}}(t)$ by $f(t)$, and $\hat{f}_{\mathrm{c}}(x)$ by $F(s)$, we get closer to the usual notation adopted in textbooks:

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

which is the definition of the **Laplace Transform** of the function $f(t)$. Notice that for each $s \in \mathbb{C}$ the above integral (if it converges) has a definite value in the complex plane, i.e. $F(s) \in \mathbb{C}$.

Laplace Transforms as Generalized Power Series VIII

- Finally, it is worth observing the difference between "Operators" and "Transforms":

$$\text{Operator (Time-domain)}: \quad u(t) \mapsto y(t),$$

$$\begin{aligned} \text{Transform}: &\quad u(t) \mapsto U(s), \\ \text{Transform}: &\quad y(t) \mapsto Y(s), \end{aligned}$$

$$\text{Operator (Frequency-domain)}: \quad U(s) \mapsto Y(s).$$

  - An **Operator** acts on the input signal and produces an output signal. **Input and output are in the same domain**.
  - A **Transform** produces a **resulting signal equivalent to the original signal** but the signals are **in different domains**.

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