A GEOMETRIC-PROBABILISTIC APPROACH TO THE INDEPENDENT COMPONENT ANALYSIS APPLIED TO AUDIO

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***Abstract* – The separation of signals when only mixtures of them are known is a Science theme which has been investigated in detail due to its large number of possible applications (e.g. speech recognition, biomedicine, signal enhancement). In order to solve this kind of problem, one of the currently most used tools is Independent Component Analysis (ICA), which has the objective of finding a separation matrix, based on signals mixtures only. In this work a hybrid simplified algorithm is introduced to find the independent component axes and carry out the separation of audio signals based on a geometrical and probabilistic analysis of the mixtures (example with images is also shown). In this algorithm, developed to be used with two signals for each iteration, one of the independent components is estimated using a geometric approach while the second is the result of an orthogonality criterion. Simulations with instantaneous mixtures resulted in measures of distortion between $-37dB$ and $-78dB$ for natural sounds and measures of $-14.7dB$ and $-22.7dB$ for images with a computational cost ($O(N)$) of $0.8MFlops$ for real time audio data acquired at $44100Hz$; and $5.0MFlops$ for monochromatic images with resolution of 640×480 pixels (one byte per pixel).**

I. INTRODUCTION

The problem of finding sources knowing only the observed mixtures composed by them without no further information about the involved signals is known as Blind Source Separation (BSS). BSS finds applications in many areas such as biomedicine [1] [2] [3] [4] [5], control [6] [7], communication systems [8] [9], economy [10], etc. In this context, audio signals represent an important class of signals to which BSS is highly desirable.

Since the neural algorithm proposed by Hérault and Jutten in 1985 [11] many works have been published about Independent Component Analysis (ICA) and Blind Source Separation (BSS). In 1994, Comon [12] gave us a great contribution introducing a mathematical analysis of the subject and defining ICA as the method of retrieving sources from a set of mixtures with the assumption that these sources have to be as independent as possible. Here, the assumption is that the mixture was made as showed in Equation (1) were $\mathbf{X}$ is the set of original sources and $\mathbf{A}$ is a square matrix that maps $\mathbf{X}$ into $\mathbf{Y}$ which is the resulting mixture

$$\mathbf{Y} = \mathbf{AX}. \tag{1}$$

The ICA can be expressed as

$$\hat{\mathbf{X}} = \mathbf{WY}, \tag{2}$$

where $\mathbf{Y}$ is the observed mixture, $\hat{\mathbf{X}}$ contains the retrieved signals and $\mathbf{W}$ is a matrix that should be equivalent to $\mathbf{A}^{-1}$ with except for permutation and scale changes [12] [13] [14]. The permutation affects the order of the resulting sources and with the scale problem we can lose information about their relative power. Another important observation is that, for the sake of simplicity, $\mathbf{A}$ is considered square and instantaneous here but, in practical situations such as audio signals, it can be different. The constraint of $\mathbf{A}$ being square means that the number of sources must be equal to the number of observed mixtures and the assert that $\mathbf{A}$ is instantaneous implies in a memoryless process where the sources, the sensors and the environmental aspects do not imply in changes in the characteristics of the mixture along the time which the samples are observed.

Comon [12], endorsed by other works [15] [16] [17] [8] [18] [19] [5] [20], suggests that the process of finding the independent sources can be divided into two distinct parts: whitening of the signals that can be accomplished using Principal Component Analysis (PCA), for instance, which guarantees the independence of cumulants up to the second order; and an orthogonal transform that maximizes the pdf's (Probability Density Function) [21] [22] entropy of the retrieved signals. The transformation must be orthogonal to keep the previous independence obtained in the first stage. Comon also claims that the problem of separating more than two sources can be accomplished in a pairwise approach [22].

In 1995, Puntonet, Prieto and Jutten published the first work about a geometrical approach to ICA [23]. This method is based on the properties of the vectorial spaces formed by sources and mixtures [14] [13]. These spaces, called mixture spaces and source spaces, are characterized by a mapping between both that is done by the matrix $\mathbf{A}$. Figure 1 shows an example for the mapping of the spaces.

The basic concept of the geometrical analysis is that the mapping caused by $\mathbf{A}$ changes the shape of the space. So, it is possible to recover $\mathbf{A}$ based on the observed mixture space only [23].

This work discuss a hybrid algorithm to extract independent sources based only on the observed mixture. It is hybrid because its kernel is based on a geometrical approach while some of the features for the separating matrix are extracted following statistical concepts.

This work starts with a short study about mixtures and how they can be modeled to provide easy algorithms for separation. The model adopted here applies to problems that can be expressed by Equation (1), for which we have an instantaneous mixture where the number of sensors is equal to the

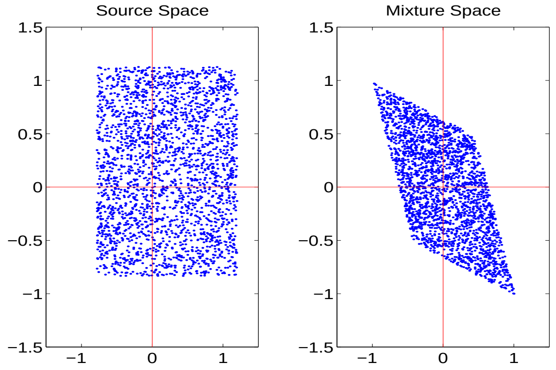

Figure 1. Geometrical analysis of independent sources and mixtures.

number of sources. This is not the case of acoustic mixtures of audio signals due to signal delays and reverberation, however, before taking into account acoustic reverberation, the case of instantaneous mixtures must be solved.

The problem will be simplified for the case of two sources and two sensors for which an algorithm will be given. A study for the extension to the cases where more than two signals are involved in the mixture will be presented. Finally, experimental results will be presented for tests made with sound signals and monochromatic images.

II. MIXTURES

ICA is strictly related to the BSS problem since by getting the independent components we can get the original sources if they are linearly independent. This could be a strong assumption, since it limits the range of mixtures that can be separated using ICA, but in Nature, it is comon to the signals to be independent. This occurs, for instance, with signals of human voice, music, natural images, biomedical signals (e.g. EEG/MEG), and many others [16].

The mixture treated here is defined by Equation (1). Let us consider the case where the size of $\mathbf{A}$ is 2×2, which means, two signals involved. In this case, Equation (1) can be written as

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad (3)$$

where k is an index for the elements of $\mathbf{y}$ and $\mathbf{x}$. Equation (3) can be split in

$$y_1(k) = a_{11}x_1(k) + a_{12}x_2(k), \quad (4)$$

$$y_2(k) = a_{21}x_1(k) + a_{22}x_2(k). \quad (5)$$

Dividing (4) by (5) leads to

$$\frac{y_1(k)}{y_2(k)} = \frac{a_{11}x_1(k) + a_{12}x_2(k)}{a_{21}x_1(k) + a_{22}x_2(k)}. \quad (6)$$

As the source signals are independent, it is probable that there exists, in the set of all elements denoted by K, a subset of samples $K^{'} = [k_1, k_2, ..., k_L]$ where one of the signals, x_1 or x_2, is near zero while the other is not. The number of elements in $K^{'}$ will depend on the Probability Density Function (PDF) of the original signals. Figure 2 shows the PDF for the beginning of the fifth("Allegro con brio") and ninth("Poco Allegro") symphonies of Beethoven. By looking at this figure it turns out apparent that, for these signals, due to the outstanding number of peaks centered in zero, the set of samples $K^{'}$ will be large. Of course this last affirmation is valid only for signals whose PDF's have a number of occurrences near zero larger than in other regions, but it has to be said that this phenomenon is valid for signals like voice, music, natural images and others. Also, it has to be noted that in some applications, e.g. communications, the PDF of the signals can be chosen so they can satisfy this characteristic.

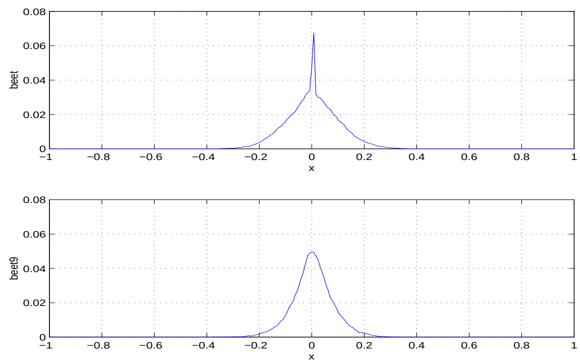

Figure 2. PDF of sample signals.

Let us examine the set of elements denoted by $K^{'}$ taking a look at the elements where $|x_1(k)| \gg 0$ and $x_2(k) \approx 0$. Equation (6) results in

$$\left.\frac{y_1(k)}{y_2(k)}\right|_{|x_1(k)|\gg 0, x_2(k)\approx 0} \approx \frac{a_{11}x_1(k)}{a_{21}x_1(k)} = \frac{a_{11}}{a_{21}}, \quad (7)$$

and, for the case where $|x_2(k)| \gg 0$ and $x_1(k) \approx 0$, Equation (6) becomes

$$\left.\frac{y_1(k)}{y_2(k)}\right|_{|x_2(k)|\gg 0, x_1(k)\approx 0} \approx \frac{a_{12}x_2(k)}{a_{22}x_2(k)} = \frac{a_{12}}{a_{22}}. \quad (8)$$

The results showed in equations (7) and (8) can be written as

$$\frac{a_{11}}{a_{21}} = \arctan \alpha, \quad (9)$$

$$\frac{a_{12}}{a_{22}} = \arctan \beta. \quad (10)$$

The mixture space for the example signals whose PDF's were presented in Figure 2 is showed in Figure 3 where it can be seen the α and β angles. To the axes formed by these angles we give the name "Visible Axes".

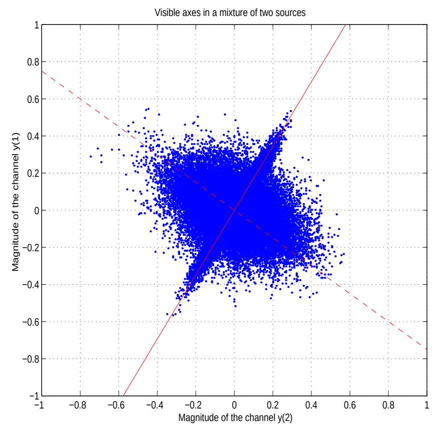

Figure 3. Space of Mixtures for the example signals.

III. THE PROBLEM FOR TWO SOURCES AND TWO MIXTURES

As showed in Section II, the problem of finding the independent components for signals that comply with some requirements proposed in that section can be simplified if the system is treated for two signals only. Now we will going to introduce a very simple algorithm for extracting the angles of the independent components for the case of two signals and, in the following section, we will show how it can be used to separate mixtures of more than two sources.

The batch algorithm for getting the independent axes, introduced here, consists of the following steps:
1. whiten the data using, for example, PCA [24];
2. divide the space of mixtures in evenly distributed sectors. The angle represented by each one of this sectors will be equal to $\theta_{sec} = \frac{\pi}{N_s}$, where N_s is the total number of sectors. Each one of these sectors is represented by an associated counter $C(s)$ $(s = -\frac{\pi}{2} : \theta_{sec} : +\frac{\pi}{2})$ that must be reseted at the start;
3. for each acquisition k, from the observed array, increment the counter associated with the sector where the pointer represented by the pair $(y_1(k), y_2(k))$ of samples $C(i) = C(i) + 1$, $i = round\left[\frac{\arctan\left(\frac{y1(k)}{y2(k)}\right)}{\theta_{sec}}\right]$ falls;
4. the counter with the greatest value at the final is related to the sector of one of the visible angles(α). To the axis denoted by this angle α we give the name *Principal Orientation Axis* (POA);
5. get the second angle using a statistical approach

$$\beta = \begin{cases} \alpha + \frac{\pi}{2} & \text{if } -\frac{\pi}{2} \le \alpha < 0, \\ \alpha - \frac{\pi}{2} & \text{if } 0 \le \alpha < \frac{\pi}{2}, \end{cases} \tag{11}$$

where α is the angle resulted from step 4.

The justification for the last step comes from the fact that the first step, the whitening of the mixtures, brings the system to an orthogonal basis that is one of the independence conditions. To keep this condition, the final transformation must be orthogonal as well. At this point, this work diverges from the work of Puntonet and Prieto [23] [14] [13] since they try to extract the second angle using geometrical methods.

If it is desired to omit the pre-whitening phase the second angle can be achieved using Equation (12) which gives the same results for the two sensors case

$$\beta = \arctan - \frac{w_{11}E[y_1^2] + w_{12}E[y_1 y_2]}{w_{12}E[y_2^2] + w_{11}E[y_1 y_2]}, \tag{12}$$

where $w_{11} = cos\alpha$, $w_{12} = -sin\alpha$ and $E[.]$ denotes expected value.

When working with more than two sources, it is better to use pre-whitening since all observed arrays can be made orthogonal at once and the task of finding the independent components is simplified. Figure 4 shows the result for the mixtures used to generate the Figure 3. Looking at Figure 4 (ranging from -90^o to $+89^o$) the POA can be find to be in the sector for the angle of -31^o while the second angle, which can not always be recognized so easily corresponds to $+56^o$.

Finally, the separation can be achieved using the transformation showed in Equation (2) where

$$\mathbf{W} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \cos\beta & -\sin\beta \end{bmatrix}. \tag{13}$$

The computational complexity of the proposed algorithm is O(N).

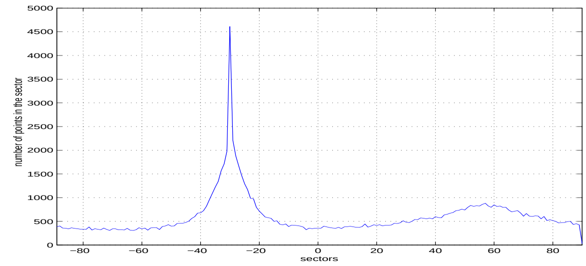

Figure 4. Result of the algorithm for the test signals.

IV. EXTENSION FOR MIXTURES INVOLVING MORE THAN TWO SOURCES

This section explains a method for applying the algorithm for two sources proposed in Section III to the case where more than two sources are present in the observed mixture.

First, let us introduce the *ICA2* block, showed in Figure 5, which is an implementation of the proposed algorithm. There are two inputs (i_1 and i_2) and two outputs ($\hat{o}_1$ and $\hat{o}_2$, where the hats mean estimates) in this block.

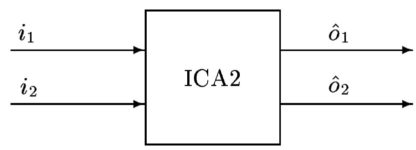

Figure 5. Stage for the separation of two sources.

Suppose that we have a problem with three signals y_1, y_2 and y_3 from three sensors which are linear combinations of three independent and supergaussian signals x_1, x_2 and x_3. Let us examine the equations related to this case. We can formulate the problem taking snapshots of the signals for each discrete instant k in a matrix form

$$\begin{bmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}. \tag{14}$$

Let us suppose that, for the three signals in this example, we have the counting of occurrences, in decreasing order, larger for $x_1(k) \approx x_2(k) \approx 0$, $|x3(k)| \gg 0$ and $x_1(k) \approx x_3(k) \approx 0$, $|x2(k)| \gg 0$ respectively. With this assumption it can be shown that applying y_1 and y_2 to the *ICA2* block results in

$$\hat{o}_1 = f_1(x_1, x_2) \tag{15}$$

$$\hat{o}_2 = f_2(x_1, x_3). \tag{16}$$

Equations (15) and (16) show that one of the signals (in this case x_1) contributed to both outputs (here f_n means a linear combination).

Now, if we apply y_1 and y_3 to another *ICA2* block, we will get $\hat{o}_{1'} = f_{1'}(x_1, x_2)$ and $\hat{o}_{2'} = f_{2'}(x_1, x_3)$ that will be a different linear combination of the signals. Next, if we apply the pairs f_1-$f_{1'}$ and f_2-$f_{2'}$ to two distinct *ICA2* blocks we will get, respectively, $\hat{x}_1$ and $\hat{x}_2$ and $\hat{x}_1$ and $\hat{x}_3$ that are the estimates of the desired signals. So, it is possible to recover the original signals with the appropriate combination of *ICA2* blocks.

Let us give a more practical example as shown in Figure (6). Suppose that we are dealing with a mixture of voices (y_1, y_2 and y_3) signals for which we have three people talking (A,

B and C). If we apply two of the mixtures, say y_1 and y_2, to the *ICA2* block we can get, for instance, $\hat{o}_1$ as being the voice of the person A mixed with the voice of the person C and $\hat{o}_2$ as being the voice of the person B mixed with the voice of the person C.

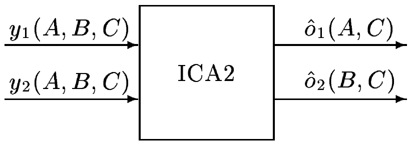

Figure 6. Example of applying signals from mixtures of three signals to the *ICA2* block. Signals A, B and C results from three people talking.

The complete separation of three signals using three sensors can be accomplished using a chain of *ICA2* blocks as shown in Figure 7. The connections needed to extract the original sources must be determined for each case. A study of how to choose these connections will be subject of future works. In our experiments, we selected them manually (hearing the sound and choosing the right outputs).

It must be noted that this approach is somehow similar to the pairwise approach introduced by Comon [12].

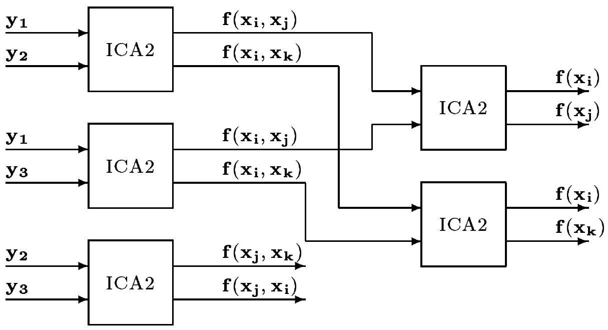

Figure 7. System for complete separation of three sources from three sensors.

V. EXPERIMENTAL RESULTS

The proposed algorithm was implemented and tested for real and simulated sound signal mixtures with known properties. All the mixtures were simulations for the simple case proposed in Equation (1) for different mixture matrices ($\mathbf{A}$). There is also an example with images whose results can be seen in Figure 11. Figure 8 shows the original images while Figure 9 shows the available mixtures (the only information known). The intermediate result after applying PCA for whitening the data can be seen in Figure 10.

The natural sound signals used were sampled at 11025Hz and the algorithm was implemented with a computational cost of $0.2MFlops$ for this signals. For CD quality sounds the computational cost is $0.8MFlops$.

The natural sound signals used were:
1. passage of the fifth symphony of Beethoven ("Allegro con brio");
2. beginning of the ninth symphony of Beethoven ("Poco Allegro");
3. female voice (counting from 1 to 10);
4. male voice (counting from 1 to 10);

The synthetic signals used were those specified in the work of Schobben et al. [25] (sine, square, saw-tooth, Gaussian noise, Cauchy, and versions of these signals with silence).

Figure 8. Original images used in the test.

Figure 9. Available mixtures.

Figure 10. Result after whitening the data with PCA.

Figure 11. Result of the *ICA2* block.

The ten mixture matrices ($\mathbf{A_n}$) used for the tests were

$$A_1 = \begin{bmatrix} -1 & -1 \\ -1 & -0.667 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -1 & -0.667 \\ -1 & -1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \qquad A_6 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} -1 & -1 \\ -1 & -0.99 \end{bmatrix} \qquad A_8 = \begin{bmatrix} -1 & -0.99 \\ -1 & -1 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} \cos(30^o) & \sin(30^o) \\ \sin(40^o) & \cos(40^o) \end{bmatrix} \qquad A_{10} = \begin{bmatrix} \cos(30^o) & \cos(40^o) \\ \sin(40^o) & \sin(30^o) \end{bmatrix}.$$

At the time we did these experiments there were no standards for the mixture matrices to be used in tests, so they were chosen by the authors hoping to catch some tricky cases like one signal much stronger than the other, one signal inverted, etc.

The performance of each test was quantified with the distortion measure proposed by Schobben et al. [25] defined as

$$D_{ij} = 10 \log_{10} \frac{E[(x_i - \hat{x}_j^{'})^2]}{E[x_i^2]}, \qquad (17)$$

where x_i is the original source and $\hat{x}_j^{'}$ is the result from the separation.

The interpretation of the result of applying Equation (17) is: large D_{ij} imply bad separation, while small values for D_{ij} imply good separation. For the sounds of voice and music used, values for $D_{ij} \leq -23dB$ represented a good separation while for $D_{ij} \leq -30dB$ the residual mixture effects were almost imperceptible (values from subjective analysis made by the authors).

The proposed method performed much better for the natural sounds. For the synthetic sounds the algorithm performed poorly and sometimes the separation was not achieved. This was expected because of the probabilistic features of the synthetic signals used (they do not show a modal distribution centered in zero).

The overall performance evaluated yielded distortions between $-37dB$ and $-78dB$ for natural sounds. For the images tested the measures of distortion were $-14.7dB$ and $-22.7dB$.

A complete description of the tests and tables with the results can be seen in [20].

VI. CONCLUSION AND REMARKS

ICA is a powerful tool that aims to find the independent components from a set of signals. So, by applying ICA to mixtures of independent sources such as audio signals we can reach some blind separation.

The algorithm proposed in this work combines a geometric-probabilistic approach. It performed well for signals with a modal distribution like real signals such as voice, music, natural images and others. Its simplicity allows its implemention in cheap microprocessors available in the market.

The extension for more than two sensors and two sources is still in work. Besides, it is possible to apply the algorithm in a pairwise approach. However, the way to choose the connections is still not clear.

The experiments made with signals of real sounds resulted in measures of distortion between $-37dB$ and $-78dB$. For the images tested and shown in this article the measures of distortion were $-14.7dB$ and $-22.7dB$. The implementation resulted in a computational cost of $0.8MFlops$ ($O(N)$) for real audio data acquired at 44100Hz and $5.0MFlops$ for monochromatic images with the resolution of 640×480 pixels (one byte per pixel).

VII. Acknowledgments

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REFERENCES

[1] Dara Ghahremani, Scott Makeig, Tzyy-Ping Jung, Anthony J. Bell, and Terrence J. Sejnowski, "Independent component analysis of simulated eeg using a three-shell spherical head model," Tech. Rep. 9601, Institute for Neural Computation, 1997.

[2] Scott Makeig, Tzyy-Ping Jung, Dara Ghahremani, and Terrence J. Sejnowski Terrence J. Sejnowski, "Independent component analysis of simulated erp data," Tech. Rep. 9606, Institute for Neural Computation, 1997.

[3] Scott Makeig, Sigurd Enghoff, Tzyy-Ping Jung, and Terrence J. Sejnowski, "Moving-window ica decomposition of eeg data reveals event-related changes in oscillatory brain activity," *ICA2000*, 2000.

[4] Scott Makeig, Anthony J. Bell, Tzyy-Ping Jung, and Terrence J. Sejnowski, "Independent component analysis of electroencephalographic data," *Advances in Neural Information Processing Systems*, 1996.

[5] A. Cichocki and A. K. Barros, "Robust batch algorithm for sequential blind extraction of noisy biomedical signals," in *ISSPA'99*, 1999.

[6] Andrew D. Back and Andrzej Cichocki, "Input variable selection using independent component analysis and higher order statistics," *Brain Science Institute, RIKEN*, 1998.

[7] Alexander Ypma and Amir Leshem, "Blind separation of machine vibration with bilinear forms," *ICA2000*, 2000.

[8] Henkjan Marsman, "A neural net approach to the source separation problem," Master's thesis, University of Twente, Department of Electrical Engineering, Aug. 1995.

[9] Y. Bar-Ness, "Bootstrapping adaptive interference cancellers: Some practical limitations.," *The Globecom Conf*, vol. 7, no. 3, pp. 1251–1255, 1982.

[10] Andrew D. Back and Andreas S. Weigend, "A first application of independent component analysis to extracting structure from stock returns," *International Journal of Neural Systems*, vol. 8, no. 5, 1997.

[11] J. Hérault, C. Jutten, and B. Ans, *Détection de grandeurs primitives dans un message composite par une architecture de calcul neuromimétique en apprentissage non supervisé*, GRETSI, 1985.

[12] Pierre Comon, "Independent Component Analysis, a new concept ?," *Signal Processing, Elsevier*, vol. 36, no. 3, pp. 287–314, Apr. 1994, Special issue on Higher-Order Statistics.

[13] Beatriz Prieto Campos, *Nuevos Algoritmos para Separación Ciega de Fuentes Utilizando Métodos Geométricos*, Tese de doutorado, Universidad de Granada, Departamento de Aquitectura y Tecnología de Computadores, July 1999.

[14] A. Prieto, B. Prieto, C. G. Puntonet, A. Cañas, and P. Martín-Smith, "Geometric separation of linear mixtures of sources: Application to speech signals," *Departamento de Arquitectura y Tecnología de Computadores*, 1998.

[15] Mark Girolami, *Self-organizing neural networks: independent component analysis and Blind source separation*, Springer, 1999.

[16] Juha Karhunen, Erkki Oja, Liuyue Wang, Ricardo Vigário, and Jyrki Joutensalo, "A class of neural networks for independent component analysis," *IEEE Transactions On Neural Networks Journal*, vol. 8, no. 3, May 1997.

[17] Jean-François Cardoso, "High-order contrasts for independent component analysis," *Neural Computation*, vol. 11, no. 1, pp. 157–192, Jan. 1999.

[18] Jean-François Cardoso, "Source separation using higher order moments," in *Proc. ICASSP*, 1989, pp. 2109–2112.

[19] Jean-François Cardoso and Beate Laheld, "Equivariant adaptive source separation," *IEEE Trans. on Sig. Proc.*, vol. 44, no. 12, pp. 3017–3030, Dec. 1996.

[20] Cristiano Rodrigues Carvalho, "Análise em componentes independentes baseada em um método geométrico probabilístico," Mestrado, Universidade Federal de Minas Gerais, Centro de Pesquisa e Desenvolvimento em Engenharia Elétrica – CPDEE, Nov. 2000.

[21] John G. Proakis, *Probability And Stochastic Processes*, Computer Science. McGraw-Hill International, 1999.

[22] Athanasios Papoulis, *Probability Random Variables, And Stochastic Processes*, Electrical Engineering. Communications and signal processing. McGraw-Hill International, 1991.

[23] C. G. Puntonet, A. Prieto, C. Jutten, M. Rodríguez-Alvarez, , and J. Ortega, "Separation of sources: a geometry-based procedure for reconstruction of n-valued signals," *Signal Processing*, vol. 46, no. 3, pp. 267–284, 1995.

[24] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge, 1985, pp. 411–455.

[25] Daniel Schobben, Kari Torkkola, and Paris Smaragdis, "Evaluation of blind signal separation methods," in *Proc. ICA99*, 1998.