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Discrete-time singular observers: $\mathcal{H}_2/\mathcal{H}_\infty$ optimality and unknown inputs

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This paper presents a family of observers which has as particular cases the 'unknown input observers' and the 'linear quadratic/$\mathcal{H}_2/\mathcal{H}_\infty$ optimal observers'. A smooth transition between these particular extreme cases through the general case is provided, corresponding to a transition of the disturbance patterns from entirely 'singular' to entirely 'regular' ones. Both 'hard' (arbitrary) and 'soft' (Gaussian or ℓ_2 signals) disturbances are allowed. The methodology presented here, although developed for discrete time systems, may be extended to the continuous time case.

1. Notation

$\mathbb{R}$ and $\mathbb{N}$ are, respectively, the real and natural numbers fields; $\mathcal{G}$ denotes either the space of zero-mean Gaussian signals or ℓ_2 signals; $\mathcal{E}(\cdot)$ is the mathematical expectation of the argument; $\rho(\cdot)$ denotes the rank of the argument; $(\cdot)^{\#}$ represents any left inverse of the argument; $\mathcal{N}(\cdot)$ is the null space of the argument; $\Im(\cdot)$ is the image space of the argument; χ_i denotes the subspace associated to the state sub-vector x_i; and $\oplus$ denotes the direct sum of subspaces.

2. Preliminaries

Consider the following linear discrete-time invariant system

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bw(k) + Eu(k) + Hf(k), \\ & \qquad\qquad\qquad\qquad\qquad x(0) = x_0 \\ y(k) &= Cx(k) + Dw(k) + Fu(k) \\ z(k) &= Tx(k) \end{aligned}\right\} \tag{1}$$

in which $x(k)\colon \mathbb{N} \to \mathbb{R}^n$ is the state vector, $y(k)\colon \mathbb{N} \to \mathbb{R}^q$ is the measurement vector, $u(k)\colon \mathbb{N} \to \mathbb{R}^r$ is a vector of known (deterministic) inputs, and $z(k)\colon \mathbb{N} \to \mathbb{R}^p$ is the signal to be estimated. The initial state condition x_0 is considered to be known and without loss of generality it can be assumed to be zero. Assume that

$$w(k) \in \mathcal{G}^m \tag{2}$$

In this sense, $w(k)\colon \mathbb{N} \to \mathbb{R}^m$, the noise signal vector, is said to be the 'soft' disturbance vector, while $f(k)\colon \mathbb{N} \to \mathbb{R}^f$ is the 'hard' disturbance vector, since there is no assumption on it. However, a 'matching condition' is required for its input matrix H. Define

$$R\colon \mathcal{N}(R) = \Im(D) \tag{3}$$

The matching condition is

$$\rho(H) = \rho(RCH) \tag{4}$$

The observability of the system (1) is assumed here. It is also assumed that all the system invariant zeros (if they exist) lie inside the open unit disc. Matrix C is also considered to be of full row rank, without loss of generality. The generic problem addressed here is the determination of an asymptotically stable filter with the structure

$$\left.\begin{aligned} \hat{x}(k+1) &= \Phi\hat{x}(k) + \sum_{i=0}^{k+n}(\Lambda_i y(i) + \Theta_i u(i)), \\ & \qquad\qquad\qquad\qquad\qquad \hat{x}(0) = 0 \\ \hat{z}(k) &= T\hat{x}(k) \end{aligned}\right\} \tag{5}$$

such that

$$\left.\begin{aligned} e(k) &= \hat{x}(k) - x(k) \\ \lim_{k\to\infty} \mathcal{E}[e(k)] &= 0 \end{aligned}\right\} \tag{6}$$

holds. This constitutes a kind of 'smoother', with possibly non-causal dynamics. For on-line estimation purposes, the structure (5) is modified as

$$\left.\begin{aligned} \hat{x}(k+1-n) &= \Phi\hat{x}(k-n) + \sum_{i=k-n}^{k}[\Lambda_i y(i) + \Theta_i u(i)], \\ & \qquad\qquad\qquad\qquad\qquad \hat{x}(0) = 0 \\ \hat{z}(k-n) &= T\hat{x}(k-n) \end{aligned}\right\} \tag{7}$$

The filter equation (7) possesses causal dynamics, since the system state vector depends only on its past values and on input signals y and u up to instant k. In fact, this formulation implies that the computation of $\hat{x}(k)$ must be performed with a time delay of $(n-1)$ units, in the worst case.

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The observer is designed in such a way that it minimizes a transfer function norm criterion (in fact, the norm minimization is guaranteed to occur only after the 'singular error' vanishes, which happens, in the worst case, after $n-1$ time intervals), for the non-causal formulation (5). The $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ norm of the transfer matrix from the disturbance input vector $w(t)$ to the estimation error $\tilde{z}(k) \triangleq z(k) - \hat{z}(k)$ are considered here. Note that the optimality of the on-line estimation of $x(k-n)$ in (7) is assured. The interpretation of $\hat{x}(k-n)$ as an estimation of $x(k)$, however, is no longer optimal. The optimal estimator for $x(k)$ would involve, necessarily, some kind of prediction.

Note that only the regular part of the output (i.e. the one which is corrupted by the noise signal) contributes to the transfer function norm. In this sense, the $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ criteria will be defined with respect to the reduced regular kernel in §5.

2.1. *Historical perspective*

The observer design techniques have long been faced with the problem of estimating the states of a system in the presence of exogenous disturbances. Two main streams have been followed up to now.

(1) *Unknown input observers*. This class of observers has been devoted to the decoupling of disturbances of arbitrary nature, entering the system state space in some specific space directions (a 'matching condition' like (4) is always necessary). The earlier references on the subject are Wang *et al.* (1975) and Kudva *et al.* (1980). Takahashi and Peres (1996) present a unified view of the field, performing a comparison between conventional unknown input and sliding mode observers.

(2) *Optimal $\mathcal{H}_2/\mathcal{H}_\infty$ filtering*. This stream of works has dealt with rather different problems, in which there is a disturbance vector entering, in principle, the whole state space and measurement space. This disturbance vector (often called 'noise') has necessarily some characterization as (2) which allows the definition of an optimization problem. The disturbances are not completely decoupled, but attenuated. The optimal attenuation is usually defined in the $\mathcal{H}_2$ (or linear quadratic) sense (Petersen and McFarlane 1994) or, more recently, also in the $\mathcal{H}_\infty$ sense (Takaba and Katayama 1996). Palhares and Peres (1998) present these optimization problems in a *Linear Matrix Inequalities* (LMIs) setting. The results therein will be employed here.

See also Hayer *et al.* (1996) for a combination of these two perspectives.

A particular vein in optimal filtering literature deals with the singular filtering problems. This is the case of problems in which some (or all) measured variables are noise-free, leading to noise-free estimates of subspaces of the state space. The $\mathcal{H}_2$ singular filtering has been addressed in Schumacher (1985), and the $\mathcal{H}_\infty$ case, more recently, in Hsu and Yu (1996) and Chevrel and Bourles (1993).

An important property of these singular problem solutions is the emergence of some directions of the state space with the complete disturbance decoupling feature. This fact has not been exploited in the literature yet, to the authors' knowledge, except in Takahashi *et al.* (1997) (by the same authors, in a continuous time setting). That property establishes a natural connection between the optimal $\mathcal{H}_2/\mathcal{H}_\infty$ observers and the unknown input observers, making them become particular cases of a more general structure. The disturbance patterns associated with each particular problem will determine the choice of a particular observer form in that general family. The present paper, in this way, rather than just combining different techniques, presents a unifying foundation for observer design.

Only the discrete-time case is approached here in order to simplify the derivations and for space reasons. The continuous time case may be dealt with through the same methodology, with differences concerning (a) hypothesis on transmission zeros and (b) the implementation structure.

3. Singular/regular canonical form

System (1) may be put into the form (see Appendix)

$$\left.\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ \mathbf{0} & A_{42} & A_{43} & A_{44} & A_{45} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{55} \end{bmatrix} \\ &\quad \times \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} w(k) \\ &\quad + \begin{bmatrix} H_{11} \\ H_{21} \\ H_{31} \\ H_{41} \\ H_{51} \end{bmatrix} g(k) + \begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \\ E_{41} \\ E_{51} \end{bmatrix} u(k) \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned}\begin{bmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ \mathbf{0} & \mathbf{0} & C_{23} & C_{24} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & C_{35} \end{bmatrix} \\ &\times \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \end{bmatrix} + \begin{bmatrix} D_{11} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} w(k) \\ &+ \begin{bmatrix} F_{11} \\ F_{21} \\ F_{31} \end{bmatrix} u(k) \end{aligned}\right\} \quad (8)$$

In the above

$$g(k) = f(k) + Vx(k) + Uw(k) \quad (9)$$

with V and U defined in (63) in the appendix. This form has the following properties:

(C1) C_{35} is square and full rank
(C2) $[C_{23} \;\; C_{24}]$ is square and full rank
(C3) D_{11} is full row rank
(C4) A_{42}, if not vanishing, has full column rank
(C5) H_{51} is square and full rank

Further decompositions may be performed in subspace $\chi_1 \oplus \chi_2 \oplus \chi_3$ if A_{42} is non-null and $[B'_{21} \;\; B'_{31} \;\; D'_{11}]'$ is not full row rank. In this case

$$\left.\begin{aligned}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \\ x_6(k+1) \\ x_7(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} \\ \mathbf{0} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} \\ \mathbf{0} & \mathbf{0} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{77} \end{bmatrix} \\ &\times \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \\ \mathbf{0} \\ B_{51} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} H_{11} \\ H_{21} \\ H_{31} \\ H_{41} \\ H_{51} \\ H_{61} \\ H_{71} \end{bmatrix} \\ &\times g(k) + \begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \\ E_{41} \\ E_{51} \\ E_{61} \\ E_{71} \end{bmatrix} u(k) \end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned}\begin{bmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & C_{17} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & C_{25} & C_{26} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & C_{37} \end{bmatrix} \\ &\times \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \end{bmatrix} + \begin{bmatrix} D_{11} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} F_{11} \\ F_{21} \\ F_{31} \end{bmatrix} u(k) \end{aligned}\right\} \quad (10)$$

Subspace $\chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_5$ may be further decomposed, if $A_{42} \neq \mathbf{0}$ and $[B'_{21} \;\; B'_{31} \;\; B'_{51} \;\; D'_{11}]'$ is not a full row rank matrix.

Define the matrices

$$\left.\begin{aligned} M_1 &= \varnothing \\ M_i &= \begin{cases} \begin{bmatrix} M_{i-1} \\ B_{i1} \end{bmatrix}, & \text{if } B_{i1} \neq \mathbf{0} \\ M_{i-1}, & \text{if } B_{i1} = \mathbf{0} \end{cases} \quad \forall i \geq 2 \end{aligned}\right\} \quad (11)$$

In the ultimate format, an additional property of the canonical form is

(C6) one of the following conditions occurs:
 (1) χ_1 vanishes
 (2) χ_1 does not vanish ($[M'_i \;\; D'_{11}]'$ is full row rank, for maximal i, or χ_2 vanishes)

In case (C6.2) there is a regular kernel in the system dynamic equations, given by subspace χ_1. In this case, regular optimal filtering problems may be defined, as stated in §5.

In case (C6.1), however, the system is fully singular. In this case, the observation error may be eliminated in a finite number of steps.

4. Observer design

Consider a system with the structure (10). This is the smallest system which exhibits the full problem complexity. In order to avoid cumbersome notation, the observer will be developed here for this system. Its generalization for systems with more blocks is straightforward.

The first steps will be devoted to the determination of the singular states. Start with subspace χ_7

$$x_7(k) = C_{37}^{-1}[y_3(k) - F_{31}u(k)] \quad (12)$$

The next step is the determination of the hard disturbance vector $g(k)$

$$g(k) = H_{71}^{-1}[x_7(k+1) - A_{77}x_7(k) - E_{71}u(k)] \quad (13)$$

Now find $\chi_6 \oplus \chi_5$

$$\begin{bmatrix} x_5(k) \\ x_6(k) \end{bmatrix} = [C_{25} \quad C_{26}]^{-1}[y_2(k) - F_{21}u(k)] \quad (14)$$

The acquired information is employed in the determination of $\chi_4 \oplus \chi_3$

$$\begin{bmatrix} x_3(k) \\ x_4(k) \end{bmatrix} = [A_{63} \quad A_{64}]^{\#} \times \left(x_6(k+1) - [A_{65} \quad A_{66} \quad A_{67}] \begin{bmatrix} x_5(k) \\ x_6(k) \\ x_7(k) \end{bmatrix} - H_{61} g(k) - E_{61} u(k) \right) \tag{15}$$

The last perfect information subspace is χ_2

$$x_2(k) = A_{42}^{\#} \times \left(x_4(k+1) - [A_{43} \quad A_{44} \quad A_{45} \quad A_{46} \quad A_{47}] \begin{bmatrix} x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \end{bmatrix} - H_{41} g(k) - E_{41} u(k) \right) \tag{16}$$

A perfect information vector $p(k)$ is defined as

$$p(k) = \left[x_2(k)' \quad x_3(k)' \quad x_4(k)' \quad x_5(k)' \quad x_6(k)' \quad x_7(k)' \quad g(k)' \quad u(k)' \right]' \tag{17}$$

Now, the soft noise corrupted information is grouped. Define a new 'measured' regularly disturbed variable:

$$q(k) = Px_1(k) + Qw(k) = \begin{bmatrix} C_{11} \\ \bar{A}_{21} \\ A_{31} \\ A_{51} \end{bmatrix} x_1(k) + \begin{bmatrix} D_{11} \\ \bar{B}_{21} \\ B_{31} \\ B_{51} \end{bmatrix} w(k)$$
$$= \begin{bmatrix} y_1(k) \\ x_2(k+1) \\ x_3(k+1) \\ x_5(k+1) \end{bmatrix} - \begin{bmatrix} C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & C_{17} & \mathbf{0} & F_{11} \\ A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & H_{21} & E_{21} \\ A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & H_{31} & E_{31} \\ A_{52} & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} & H_{51} & E_{51} \end{bmatrix} p(k) \tag{18}$$

In the above, $\bar{B}_{21}$ means a full row rank partition of B_{21} (possibly subject to a coordinate transformation) and $\bar{A}_{21}$ its corresponding rows in A_{21}. The regularly disturbed filtering problem is defined as the search for an optimal filtering gain matrix L

$$\hat{x}_1(k+1) = (A_{11} - LP)\hat{x}_1(k) + Lq(k) + E_r p(k) \tag{19}$$

in which

$$E_r \triangleq [A_{12} \quad A_{13} \quad A_{14} \quad A_{15} \quad A_{16} \quad A_{17} \quad H_{11} \quad E_{11}]$$

Equations (12)–(16) express the algebraic connection of the perfect measurements (y_2, y_3) and deterministic inputs u with the state estimates $(\hat{x}_2 \ldots \hat{x}_7)$. The dynamic dependence of x_1 with (y_1, y_2, y_3) and u is expressed in equations (17)–(19). The general structure (7) is embedded implicitly in these equations.

5. Filtering scheme

In the case of χ_1 not vanishing, the reduced order system obtained from the decomposition procedure is given by

$$\left.\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + B_{11}w(k) + E_r p(k), \\ & \qquad x_1(0) = x_0 \\ q(k) &= Px_1(k) + Qw(k) \\ z_r(k) &= T_r x_1(k) \end{aligned}\right\} \tag{20}$$

in which $x_1(k)\colon \mathbb{N} \to \mathbb{R}^{n_r}$ is the reduced-order state vector, $w(k)\colon \mathbb{N} \to \mathbb{R}^m$ is the (full-order) exogenous 'soft disturbance' input vector, $q(k)\colon \mathbb{N} \to \mathbb{R}^l$ is the extended measurement output vector, $z_r(k)\colon \mathbb{N} \to \mathbb{R}^{p_r}$ is a linear combination of the reduced-order state variables to be estimated and $p(k)\colon \mathbb{N} \to \mathbb{R}^{r_r}$ is a known deterministic input vector including the deterministic input vector $u(k)$, the hard disturbance vector $g(k)$ (defined in (9)) and also the singularly determined states $x_j(k)$, $j = 2, \ldots, b$. The amount n_r corresponds to the dimension of χ_1.

The aim is to determine an asymptotically stable linear filter of order n_r given by

$$\left.\begin{aligned} \hat{x}_1(k+1) &= A_{11}\hat{x}_1(k) + L(q(k) - P\hat{x}_1(k)) + E_r p(k), \\ & \qquad \hat{x}_1(0) = 0 \\ \hat{z}_r(k) &= T_r \hat{x}_1(k) \end{aligned}\right\} \tag{21}$$

where $L \in \mathbb{R}^{n_r \times l}$ is an unknown matrix to be determined. Considering $e(k) \triangleq x_1(k) - \hat{x}_1(k)$ and $\tilde{z}_r(k) \triangleq z_r(k) - \hat{z}_r(k)$, the estimation error dynamics is given by

$$\left.\begin{aligned} e(k+1) &= A_\psi e(k) + B_\psi w(k), \quad e(0) = x_1(0) - \hat{x}_1(0) \\ \tilde{z}_r(k) &= T_r e(k) \end{aligned}\right\} \tag{22}$$

where $A_\psi \triangleq (A_{11} - LP), B_\psi \triangleq (B_{11} - LQ)$. Three optimal filter schemes are presented in the next subsections in terms of LMIs: the optimal $\mathcal{H}_2$, the optimal $\mathcal{H}_\infty$ and the central $\mathcal{H}_\infty$ ones. The correspondence of these LMIs with the classical $\mathcal{H}_2/\mathcal{H}_\infty$ filtering results can be found in Palhares and Peres (1998).

5.1. *Optimal $\mathcal{H}_2$ Filtering*

Suppose that the power spectral density matrix of w is known, with joint covariance matrix given by

$$\mathcal{E}\{w(k)w(k)'\} = \begin{bmatrix} B_{11}B_{11}' & B_{11}Q' \\ QB_{11}' & QQ' \end{bmatrix}$$

The regular $\mathcal{H}_2$ filter design is concerned with the minimization of the estimation error variance given by

$$\lim_{k\to\infty} \mathcal{E}\{\tilde{z}_r'\tilde{z}_r\} \tag{23}$$

If A_ψ is asymptotically stable, the estimation error variance (23) is asymptotically bounded by

$$\mathrm{Tr}\{B_\psi'\Gamma B_\psi\} \tag{24}$$

where $\Gamma = \Gamma' \geq \mathbf{0}, \Gamma \in \mathbb{R}^{n_r\times n_r}$ solves the Lyapunov equation

$$\mathcal{R}(\Gamma) = \mathbf{0} \tag{25}$$

with

$$\mathcal{R}(\Gamma) \triangleq A_\psi'\Gamma A_\psi - \Gamma + T_r'T_r \tag{26}$$

The next theorem states, in terms of LMIs, an optimization procedure whose optimal solution yields the optimal $\mathcal{H}_2$ filter gain.

Theorem 1: *The optimal solution of*

$$\min_{J,Y,W} \mathrm{Tr}\{J\} \tag{27}$$

subject to

$$\begin{bmatrix} J & B_{11}'Y - Q'W' \\ YB_{11} - WQ & Y \end{bmatrix} \geq \mathbf{0} \tag{28}$$

$$\begin{bmatrix} Y & YA_{11} - WP & \mathbf{0} \\ A_{11}'Y - P'W' & Y & T_r' \\ \mathbf{0} & T_r & \mathbf{I} \end{bmatrix} \geq \mathbf{0} \tag{29}$$

$$Y > \mathbf{0} \tag{30}$$

with $Y = Y' \in \mathbb{R}^{n_r\times n_r}$, $W \in \mathbb{R}^{n_r\times q}$ *and* $J = J' \in \mathbb{R}^{m\times m}$ *is such that* $\mathrm{Tr}\{J\} = \lim_{k\to\infty}\mathcal{E}\{\tilde{z}_r'\tilde{z}_r\}$ *and*

$$L = Y^{-1}W$$

is the optimal $\mathcal{H}_2$ filter gain.

Proof:

(Sufficiency) Suppose that there exist $Y = Y' > \mathbf{0}$ and W satisfying (29). Then, by a standard Schur complement argument, (29) is equivalent to

$$(YA_{11} - WP)'Y^{-1}(YA_{11} - WP) - Y + T_r'T_r \leq \mathbf{0} \tag{31}$$

or

$$(A_{11} - Y^{-1}WP)'Y(A_{11} - Y^{-1}WP) - Y + T_r'T_r \leq \mathbf{0} \tag{32}$$

implying that $L \triangleq Y^{-1}W$ and $\Gamma \triangleq Y$ are feasible solutions to $\mathcal{R}(\Gamma) \leq \mathbf{0}$, where $A_\psi = A_{11} - LP$, and hence (21) is asymptotically stable.

(Necessity) Consider a pair of matrices (Γ, L) satisfying (25), thus implying $\mathcal{R}(\Gamma) \leq \mathbf{0}$ (note that it is equivalent to suppose system (21) as being asymptotically stable), then after some algebraic manipulations, one gets

$$(\Gamma A_{11} - \Gamma LP)'\Gamma^{-1}(\Gamma A_{11} - \Gamma LP) - \Gamma + T_r'T_r \leq \mathbf{0} \tag{33}$$

following that $W \triangleq \Gamma L$ and $Y = \Gamma$ satisfy the inequality (29).

From the objective function and applying the Schur complement to inequality (28) with $Y > \mathbf{0}$, one gets

$$\begin{aligned}\mathrm{Tr}\{J\} &\geq \mathrm{Tr}\{(YB_{11} - WQ)'Y^{-1}(YB_{11} - WQ)\} \\ &= \mathrm{Tr}\{(B_{11} - Y^{-1}WQ)'Y(B_{11} - Y^{-1}WQ)\} \\ &= \mathrm{Tr}\{B_\psi'YB_\psi\} \geq \lim_{k\to\infty}\mathcal{E}\{\tilde{z}_r'\tilde{z}_r\}\end{aligned} \tag{34}$$

In fact, since no other constraint is imposed to the variable J, the minimization of the linear cost (27) ensures that

$$\mathrm{Tr}\{J\} = \mathrm{Tr}\{(YB_{11} - WQ)'Y^{-1}(YB_{11} - WQ)\}$$

Furthermore, since that (Γ, L) satisfy (24)–(25), then $Y = \Gamma$, $W = LY$ and $J = (YB_{11} - WQ)'Y^{-1}(YB_{11} - WQ)$ are feasible solutions to the above optimization problem, implying that

$$\mathrm{Tr}\{J\} = \lim_{k\to\infty}\mathcal{E}\{\tilde{z}_r'\tilde{z}_r\}$$

holds. □

5.2. *Optimal $\mathcal{H}_\infty$ filtering*

Supposing that the input disturbance has unknown spectrum with $w \in \ell_2[0;\infty)$, the optimal regular $\mathcal{H}_\infty$ filtering design can be used. In this case, the problem is to determine a stable filter of the form (21) in order to ensure the minimum disturbance attenuation level γ such that

$$\|H_{\tilde{z}_r w}\|_\infty = \sup_{0\neq w\in\ell_2[0;\infty)} \frac{\|\tilde{z}_r\|_2}{\|w\|_2} < \gamma$$

where $H_{\tilde{z}_r w}$ denotes the transfer function from the noise signal w to the output $\tilde{z}_r$. As it is well known (de Souza and Xie 1992), with A_ψ asymptotically stable, $\|H_{\tilde{z}_r w}\|_\infty < \gamma$ if and only if the following Riccati filtering equation

$$\begin{aligned}A_\psi'XA_\psi - X + T_r'T_r + \gamma^{-2}A_\psi'XB_\psi \\ \times(\mathbf{I} - \gamma^{-2}B_\psi'XB_\psi)^{-1}B_\psi XA_\psi \leq \mathbf{0}\end{aligned} \tag{35}$$

admits a positive definite matrix X as its solution. The optimal $\mathcal{H}_\infty$ gain can be obtained through the following optimization procedure.

Theorem 2: *The optimal solution of*

$$\min_{Y,W,\delta} \delta \tag{36}$$

subject to

$$\begin{bmatrix} Y & \mathbf{0} & A_{11}'Y - P'W' & T_r' \\ \mathbf{0} & \delta\mathbf{I} & B_{11}'Y - Q'W' & \mathbf{0} \\ YA_{11} - WP & YB_{11} - WQ & Y & \mathbf{0} \\ T_r & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \geq \mathbf{0} \tag{37}$$

$$Y > \mathbf{0} \tag{38}$$

where $\delta \triangleq \gamma^2$ *and* $Y = Y' \in \mathbb{R}^{n_r \times n_r}$, $Z \in \mathbb{R}^{n_r \times q}$ *is such that*

$$\|H_{z_r w}\|_\infty = \sqrt{\delta} \tag{39}$$

and $L = Y^{-1}W$ *is the optimal* $\mathcal{H}_\infty$ *filter gain.*

Proof: First, the equivalence between a triple (Y, W, δ) satisfying (37)–(38) and (X, L, γ) solution of (35) is shown.

(Sufficiency) Using Schur's complement, (37) can be rewritten as

$$\begin{bmatrix} YA_{11} - WP & YB_{11} - WQ \\ T_r & \mathbf{0} \end{bmatrix}' \begin{bmatrix} Y^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \times \begin{bmatrix} YA_{11} - WP & YB_{11} - WQ \\ T_r & \mathbf{0} \end{bmatrix} - \begin{bmatrix} Y & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{I} \end{bmatrix} \leq \mathbf{0} \tag{40}$$

where $\delta \triangleq \gamma^2$. It is straightforward to rewrite (40) as

$$\begin{bmatrix} A_{11} - Y^{-1}WP & B_{11} - Y^{-1}WQ \\ T_r & \mathbf{0} \end{bmatrix}' \begin{bmatrix} Y & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \times \begin{bmatrix} A_{11} - Y^{-1}WP & B_{11} - Y^{-1}WQ \\ T_r & \mathbf{0} \end{bmatrix} - \begin{bmatrix} Y & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I} \end{bmatrix} \leq \mathbf{0} \tag{41}$$

implying, by a standard Schur complement argument, that $L \triangleq Y^{-1}W$, $X \triangleq Y$ and γ solve (35), ensuring that $\|H_{z_r w}\|_\infty \leq \gamma$.

(Necessity) Suppose that, for a fixed $\gamma > 0$, there exists L such that $\|H_{z_r w}\|_\infty \leq \gamma$ and thus $X > \mathbf{0}$ satisfying (35), it follows from Schur complement and some algebraic manipulations that (35) is equivalent to

$$\begin{bmatrix} YA_{11} - YLP & YB_{11} - YLQ \\ T_r & \mathbf{0} \end{bmatrix}' \begin{bmatrix} Y^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \times \begin{bmatrix} YA_{11} - YLP & YB_{11} - YLQ \\ T_r & \mathbf{0} \end{bmatrix} - \begin{bmatrix} Y & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I} \end{bmatrix} \leq \mathbf{0} \tag{42}$$

implying that $W \triangleq YL$, $Y \triangleq X$ and $\delta \triangleq \gamma^2$ are feasible solutions to (37).

Furthermore, since the optimization problem is convex (linear criterion under linear constraints), the optimal solution (Y, W, δ) yields $L = Y^{-1}W$, such that $\min \|H_{z_r w}\|_\infty = \gamma^* = \sqrt{\delta^*}$. □

5.3. *Central $\mathcal{H}_\infty$ filtering*

The central $\mathcal{H}_\infty$ filtering design deals with the problem of determining a stable filter (21) which guarantees the minimization of an upper bound to the $\mathcal{H}_2$ performance criterion and satisfies the bound $\|H_{z_r w}\|_\infty < \gamma$ for a given γ. In other words, this filter design (also called the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter design) claims to achieve a compromise between both performance criteria.

Theorem 3: *The optimal solution of*

$$\min_{J, Y, W} \operatorname{Tr}\{J\} \tag{43}$$

subject to

$$\begin{bmatrix} J & B_{11}'Y - Q'W' \\ YB_{11} - WQ & Y \end{bmatrix} \geq \mathbf{0} \tag{44}$$

$$\begin{bmatrix} Y & \mathbf{0} & A_{11}'Y - P'W' & T_r' \\ \mathbf{0} & \gamma^2\mathbf{I} & B_{11}'Y - Q'W' & \mathbf{0} \\ YA_{11} - WP & YB_{11} - WQ & Y & \mathbf{0} \\ T_r & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \geq \mathbf{0} \tag{45}$$

$$Y > \mathbf{0} \tag{46}$$

with $Y = Y' \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times r}$ *and* $J = J' \in \mathbb{R}^{m \times m}$ *is such that*

$$\operatorname{Tr}\{J\} \geq \|H_{z_r w}\|_2^2; \qquad \|H_{z_r w}\|_\infty \leq \gamma \tag{47}$$

and the filtering gain is given by

$$L = Y^{-1}W \tag{48}$$

Proof: Firstly, suppose that for $\gamma > 0$ given, there exists (Y, W) solution of (45)–(46). Applying the Schur complement, it follows (see Theorem 2) that (45) is equivalent to (35), implying that $L \triangleq Y^{-1}W$, $X \triangleq Y$ and γ solve (35), ensuring $\|H_{z_r w}\|_\infty \leq \gamma$.

Since the inequality (45) can be rewritten as (35) such that $L \triangleq Y^{-1}W$, $X \triangleq Y$, the following inequality, derived from map (26),

$$\mathcal{R}(X) \leq -\gamma^{-2}A_\psi' XB_\psi(\mathbf{I} - \gamma^{-2}B_\psi' XB_\psi)^{-1}B_\psi XA_\psi \tag{49}$$

holds, and since the system is observable with $X > \mathbf{0}$, it follows from de Souza and Xie (1992, Theorem 2.1) that $\mathbf{I} - \gamma^{-2}B_\psi' XB_\psi > \mathbf{0}$, thus $\mathcal{R}(X) \leq \mathbf{0}$ and hence, $X \geq \Gamma$ in (25), such that (47) holds. □

6. Numerical example

An academic example is provided, in order to illustrate the synthesis method developed here. No *a priori* known inputs will be considered. Consider the system:

$$\left.\begin{aligned}
A &= \begin{bmatrix} 0.0381 & 0.2347 & 0.1938 & -0.3096 & 0.3422 & 0.1206 & 0.1151 \\ -0.4873 & 0.0045 & 0.0504 & -0.1326 & -0.0624 & -0.1428 & 0.0318 \\ 0 & 0 & 0.4582 & 0.1222 & -0.0787 & 0.4735 & 0.0320 \\ -0.3341 & 0.3333 & 0.7975 & 0.2134 & 0.1174 & 0.0341 & 0.0707 \\ 0 & 0 & 0.4458 & -0.5110 & 0.3513 & 0.8987 & -0.1885 \\ 0 & 0 & 0.0766 & 0.3650 & 0.1159 & -0.0304 & 0.1672 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2722 \end{bmatrix} \\
B &= \begin{bmatrix} 0.2666 & 0.2467 & 0.6940 & 0.8171 \\ 0.9701 & 0.8440 & 0.4558 & 0.0221 \\ 0 & 0 & 0 & 0 \\ 0.1607 & 0.7078 & 0.5824 & 0.9915 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
C &= \begin{bmatrix} 0.6762 & 0.9448 & 0.4604 & 0.6056 & 0.3438 & 0.0150 & 0.7830 \\ 0 & 0 & 0 & 0 & 0.6701 & 0.8292 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5228 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2742 \end{bmatrix} \\
D &= \begin{bmatrix} 0.7467 & 0.4364 & 0.0063 & 0.5848 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
H &= [0.7539 \quad 0.1712 \quad 0.0043 \quad 0.4957 \quad 0.0799 \quad 0.1631 \quad 0.9033]'
\end{aligned}\right\} \tag{50}$$

The system is already presented in the decomposed form. This system has one regularly disturbed measurement variable, y_1, and three singular measurement variables, y_2, y_3 and y_4. The state variables x_5, x_6 and x_7 are directly obtainable from these singular measurements. From the equation of $x_7(k+1)$ comes the determination of the hard disturbance variable $g(k)$ defined in (9). From x_5 and x_6 equations, state variables x_3 and x_4 are also directly extracted. The equation of $x_4(k+1)$ may not furnish exact information, since it is corrupted by noise. Although the equation of $x_3(k+1)$ is noise-free, it is unusable for the purpose of getting further exact information about the state vector too, because $A_{32} = 0$. The system regular kernel is, therefore, composed by states x_1 and x_2. The regularly corrupted available information sources are the equations of y_1 and $x_4(k+1)$. Note that the equation of $x_3(k+1)$ is also unusable for the purpose of furnishing noise corrupted information, since it is noise-free. Note also that matrix $[B_{11}' \; B_{21}' \; B_{41}' \; D_{11}']'$ is square and full rank, which means there is no further information redundancy to be exploited.

Vector $p(k)$, defined in (17), is obtained from

$$p(k-1) = \begin{bmatrix} 2.6984 & 0.5623 & -2.2503 & -1.3858 & -0.7631 & 0.3119 \\ -0.5663 & 5.1225 & -1.3319 & -0.1830 & 1.0710 & -1.2450 \\ 0 & 0 & 0 & 1.4923 & -2.3669 & 0 \\ 0 & 0 & 0 & 0 & 1.9128 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.6470 \\ 0 & 0 & 4.0374 & 0 & 0 & -1.0990 \end{bmatrix} \times \begin{bmatrix} y_2(k) \\ y_3(k) \\ y_4(k) \\ y_2(k-1) \\ y_3(k-1) \\ y_4(k-1) \end{bmatrix} \tag{51}$$

This matrix equation synthesizes the observer design equations (12)–(16). The noisy information vector $q(k)$, defined in (18) is given by

$$q(k-1) = \begin{bmatrix} y_1(k-1) \\ p_2(k) \end{bmatrix} - \begin{bmatrix} 0.4604 & 0.6056 & 0.3438 & 0.0150 & 0.7830 & 0 \\ 0.7975 & 0.2134 & 0.1174 & 0.0341 & 0.0707 & 0.4957 \end{bmatrix} \times p(k-1) \tag{52}$$

The reduced order regular kernel estimator, following (19), is given by

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.0381 & 0.2347 \\ -0.4873 & 0.0045 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} + L\left(q(k) - \begin{bmatrix} 0.6762 & 0.9448 \\ -0.3341 & 0.3333 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} \right) + \begin{bmatrix} 0.1938 & -0.3096 & 0.3422 & 0.1206 & 0.1151 & 0.7539 \\ 0.0504 & -0.1326 & -0.0624 & -0.1428 & 0.0318 & 0.1712 \end{bmatrix} \times p(k) \tag{53}$$

The noise $w(k)$ enters the reduced-order system through matrices

$$\left.\begin{aligned} B_{11} &= \begin{bmatrix} 0.2666 & 0.2467 & 0.6940 & 0.8171 \\ 0.9701 & 0.8440 & 0.4558 & 0.0221 \end{bmatrix} \\ Q &= \begin{bmatrix} 0.7467 & 0.4364 & 0.0063 & 0.5848 \\ 0.1607 & 0.7078 & 0.5824 & 0.9915 \end{bmatrix} \end{aligned}\right\} \tag{54}$$

The filtering weighting matrix T_r is chosen as:

$$T_r = [1 \quad 0] \tag{55}$$

In this way, the algorithm of central $\mathcal{H}_\infty$ filtering with the bound $\|H_{\tilde{z}_r w}\|_\infty \le \gamma = 0.7305$ leads to the constant filtering gain matrix:

$$L = \begin{bmatrix} 0.0316 & 0.7273 \\ 0.2428 & 0.3731 \end{bmatrix} \tag{56}$$

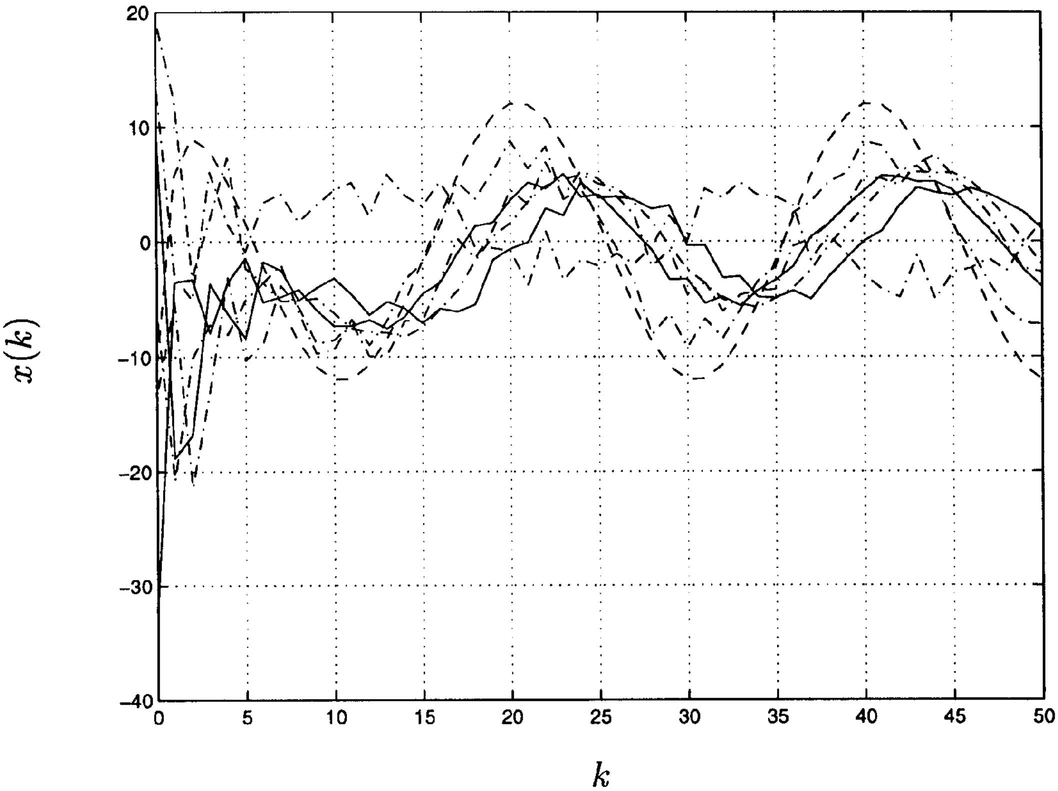

Figure 1. State vector evolution.

The minimal upper bound to the $\mathcal{H}_2$ norm is found to be $\mathrm{Tr}\{J\} = 0.1995$.

The observer design is complete. A simulation is now performed in order to demonstrate the resulting observer features. Let $g(k)$ be given by

$$g(k) = 10 \sin\left(\frac{\pi k}{10} + 1.23\right) \tag{57}$$

The initial state vector is randomly chosen as:

$$x'(0) = [19.3389\ 13.6889\ -6.1921\ -13.4566\ 9.3444\ -32.2866\ -10.7033] \tag{58}$$

The input vector $w(k)$ is chosen to be zero mean white noise with identity covariance matrix. The system states evolution is shown in figure 1. The state estimation error e_r of the regular sub-system $\left[x_1'\ x_2'\right]'$ and e_s of the singular sub-system $\left[x_3'\ \cdots\ x_7'\right]'$ are shown in figure 2 top and bottom, respectively.

Alternatively, $\mathcal{H}_2$ or $\mathcal{H}_\infty$ synthesis could be used for the regular reduced order system filter design, following the results of Theorems 1 and 2, leading to different filter gain matrices.

7. Conclusions

In this paper, the intrinsic relationship between the disturbance patterns considered in a system model and the structure of optimal observers for the system state vector has been investigated. It has been found that different disturbance patterns lead to different specific observer structures.

Only discrete time systems have been considered, but the same methodology could be employed for the continuous-time case.

Two kinds of disturbances have been dealt with: the 'hard' disturbances, of entirely arbitrary nature, and the 'soft' ones, which receive some particular characterization (as being Gaussian or belonging to ℓ_2). Due to their nature, the hard disturbances must be decoupled, while the soft disturbances may be only attenuated. These tasks are both accomplished by the proposed observer design. Additionally, the problem singularity may lead to more disturbance decoupling directions, which contain only soft disturbances.

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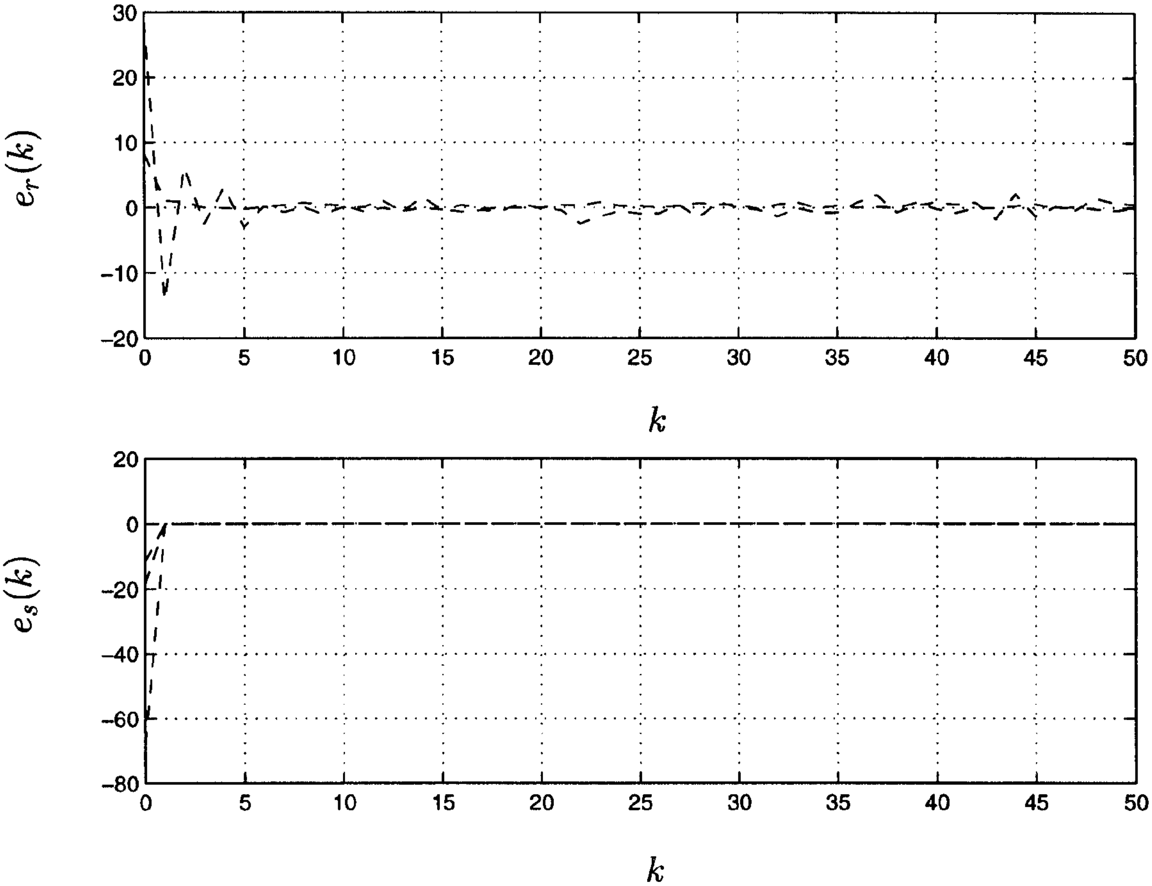

Figure 2. State estimation error: regular states (top) and singular states (bottom).

Appendix: regular/singular decomposition

This appendix describes the algorithm proposed for system decomposition into a singular part and a regular part. The 'regular' sub-system is associated with the subspace in which the noise and uncertainties corrupt the state estimates. The 'singular' sub-system is associated with the noise and uncertainty-free subspace. In this way, the singular sub-system allows the deterministic extraction of state trajectories from the available measurements. The system equations (1) are initially rewritten:

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bw(k) + Hf(k) + Eu(k), \\ &\qquad\qquad\qquad\qquad x(0) = x_0 \\ y(k) &= Cx(k) + Dw(k) + Fu(k) \end{aligned}\right\} \quad (59)$$

Part I: hard disturbances isolation

The algorithm first part makes the reduction of the full problem to a problem without 'hard disturbances'.

Step 1. Matrix D is singular (in the sense it does not corrupt at least one measurement direction) if and only if it has row rank deficiency. Remember also that C has full row rank. This means that there are coordinate transformations in y and x vectors such that:

$$\left.\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} w(k) \\ &\quad + \begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix} f(k) + \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} u(k) \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} \\ \mathbf{0} & C_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} D_{11} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} u(k) \end{aligned}\right\} \quad (60)$$

Let $\ell = \rho(D_{11})$ denote the number of regularly disturbed outputs, and $s = m - \ell$ denote the number of disturbance-free (singularly measured) outputs. In system (60) sub-vector $y_2 \in \mathbb{R}^s$. A corresponding partition of x has been defined such that $x_2 \in \mathbb{R}^s$, which defines the partition in the dynamic equation.

After this step, the relevant full-rank blocks are:

- $D_{11} \rightarrow$ full row rank
- $C_{11} \rightarrow$ full row rank
- $C_{22} \in \mathbb{R}^{s \times s} \rightarrow$ full row and column rank
- $H_{21} \rightarrow$ full column rank. This comes from the matching condition hypothesis.

Step 2. Now transform the basis of sub-vector x_2, in order to obtain a square non-singular H_{31}

$$\left.\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \\ &+ \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{bmatrix} w(k) + \begin{bmatrix} H_{11} \\ \mathbf{0} \\ H_{31} \end{bmatrix} f(k) \\ &+ \begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \end{bmatrix} u(k) \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ \mathbf{0} & C_{22} & C_{23} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \\ &+ \begin{bmatrix} D_{11} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} u(k) \end{aligned}\right\} \quad (61)$$

After this step, the relevant full-rank blocks are

- $D_{11} \rightarrow$ full row rank
- $C_{11} \rightarrow$ full row rank
- $[C_{22} \ \ C_{23}] \rightarrow$ full row and column rank
- $H_{31} \rightarrow$ full row and column rank

Step 3. Define now the variable:

$$d(k) = H_{31}^{-1}[x_3(k+1) - A_{33}x_3(k)] \quad (62)$$

Note that this variable is, in principle, exactly known from the uncorrupted measurement y_2. Variable $f(k)$ is written as:

$$f(k) = d(k) - H_{31}^{-1}[A_{31}x_1(k) + A_{32}x_2(k) + B_{31}w(k)] \quad (63)$$

The system equation becomes:

$$\left.\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} - H_{11}H_{31}^{-1}A_{31} & A_{12} - H_{11}H_{31}^{-1}A_{32} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \mathbf{0} & \mathbf{0} & A_{33} \end{bmatrix} \\ &\times \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} B_{11} - H_{11}H_{31}^{-1}B_{31} \\ B_{21} \\ \mathbf{0} \end{bmatrix} w(k) \\ &+ \begin{bmatrix} H_{11} \\ \mathbf{0} \\ H_{31} \end{bmatrix} d(k) + \begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \end{bmatrix} u(k) \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ \mathbf{0} & C_{22} & C_{23} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \\ &+ \begin{bmatrix} D_{11} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} u(k) \end{aligned}\right\} \quad (64)$$

Step 4. Take now only the sub-vector $\begin{bmatrix} x_1' & x_2' \end{bmatrix}'$ equations:

$$\left.\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} - H_{11}H_{31}^{-1}A_{31} & A_{12} - H_{11}H_{31}^{-1}A_{32} \\ A_{21} & A_{22} \end{bmatrix} \\ &\times \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_{11} - H_{11}H_{31}^{-1}B_{31} \\ B_{21} \end{bmatrix} w(k) \\ &+ \begin{bmatrix} H_{11} & A_{13} & E_{11} \\ \mathbf{0} & A_{23} & E_{21} \end{bmatrix} \begin{bmatrix} d(k) \\ x_3(k) \\ u(k) \end{bmatrix} \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} \\ \mathbf{0} & C_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} D_{11} \\ \mathbf{0} \end{bmatrix} w(k) \\ &+ \begin{bmatrix} \mathbf{0} & C_{13} & F_{11} \\ \mathbf{0} & C_{23} & F_{21} \end{bmatrix} \begin{bmatrix} d(k) \\ x_3(k) \\ u(k) \end{bmatrix} \end{aligned}\right\} \quad (67)$$

Note that all 'deterministic inputs' have been grouped into a single vector. In this way, the system decomposition has fallen into a system like (59), with $H = \mathbf{0}$.

Part II: regular/singular decomposition

The second part of the algorithm provides the determination of the system 'regular kernel', if it exists. Start with system in the form

$$\left.\begin{aligned}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} w(k) + \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} u(k) \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} \\ \mathbf{0} & C_{22} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} D_{11} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} u(k)\end{aligned}\right\} \tag{66}$$

Let $\ell = \rho(D_{11})$ denote the number of regularly disturbed outputs, and $s = m - \ell$ denote the number of the remaining disturbance-free (singularly measured) outputs, after part I. In system (66) the sub-vector $y_2 \in \mathbb{R}^s$. A corresponding partition of x has been defined such that $x_2 \in \mathbb{R}^s$, which defines the partition in the dynamic equation.

The relevant full-rank blocks are

- $D_{11} \rightarrow$ full row rank
- $C_{11} \rightarrow$ full row rank
- $C_{22} \in \mathbb{R}^{s \times s} \rightarrow$ full row and column rank

If $[B_{21}' \ D_{11}']'$ is full row rank, there is nothing to do. The algorithm stops, and the regular kernel is χ_1, with measurement vector $[y_1' \ x_2']'$. Otherwise, the algorithm part II must be performed.

Step 1. Decompose B_{21}, finding a full row rank (note that the state variables may be combined with some component of the noisy output measurement vector) $[B_{21}' \ D_{11}']'$ in $[B_{21}' \ \mathbf{0} \ D_{11}']'$

$$\left.\begin{aligned}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} B_{11} \\ B_{21} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \end{bmatrix} u(k) \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ \mathbf{0} & C_{22} & C_{23} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} D_{11} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} u(k)\end{aligned}\right\} \tag{67}$$

If $\rho(A_{31}) = 0$, then the decomposition stops here. In such a case, there is no means of getting further exact information about x_1, and the regular kernel is sub-space χ_1 with measurement vector y_1. Otherwise, go to step 2.

Step 2. As $\rho(A_{31}) \neq 0$, there is a decomposition of x_1 in which

$$\left.\begin{aligned}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ \mathbf{0} & A_{42} & A_{43} & A_{44} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \\ E_{41} \end{bmatrix} u(k) \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ \mathbf{0} & \mathbf{0} & C_{23} & C_{24} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} D_{11} \\ \mathbf{0} \end{bmatrix} w(k) + \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} u(k)\end{aligned}\right\} \tag{68}$$

After this step, A_{42} is full column rank (by construction) and sub-vector x_2 is completely determinable from $x_4(k+1)$ equation.

At this point, if $[B_{21}' \ B_{31}' \ D_{11}']'$ is full row rank, the algorithm stops and the system regular kernel is χ_1 with measurement vector y_1.

If $\dim(x_1) = 0$, the algorithm also stops, and there is no regular kernel in the system.

If $\dim(x_1) \neq 0$ and $[B_{21}' \ B_{31}' \ D_{11}']'$ is not full row rank, take the sub-system given by $\chi_1 \oplus \chi_2 \oplus \chi_3$ and return to step 1. As the dimension of the considered subspace is strictly smaller at each iteration, the algorithm stops after a finite number of iterations.

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