FEDERAL UNIVERSITY OF MINAS GERAIS

PHD THESIS

Robust Stabilization of Uncertain Nonlinear Systems using Differential Algebraic Representations

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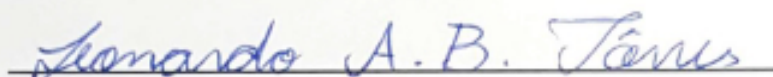
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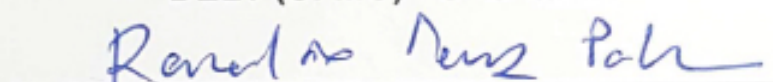
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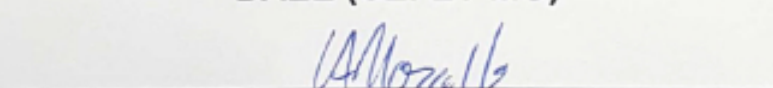
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Abstract

Regional robust stabilization for a class of uncertain MIMO nonlinear systems with parametric uncertainties is investigated. The closed-loop robust stability is achieved using linear time-invariant state feedback control. In this context, two cases are investigated: (i) uncertain nonlinear systems resulting from attempts to use the well-known Input-Output Feedback Linearization technique applied considering nominal parameters; and (ii) uncertain nonlinear systems with input saturation. In both cases, the fact that the uncertain systems have Differential Algebraic Representations (DAR) is the main theoretical assumption employed to derive sufficient conditions, in the form of Linear Matrix Inequalities (LMI), to solve the corresponding control problem. The regional character of the stability result obtained using this approach is associated with the largest ellipsoidal Domain of Attraction (DOA), considered to be inside a given polytopic region in the closed-loop system state space, which is a byproduct of solving the associated optimization problem of searching for appropriate feedback gain matrices. Specifically, the thesis contributions are new sufficient LMI conditions with new decision variables used to compute the feedback gain matrices without prior knowledge of an initial stabilizing matrix. The new conditions have also shown favorable comparisons with recently published similar control design methodologies, particularly for the case of uncertain nonlinear systems with input saturation, where a polytopic description of this nonlinearity has led to new LMI conditions.

Acknowledgements

All Praise Is Due to God, Lord of The Worlds. "Quran 1:2"

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Abbreviations

LMI	**L**inear **M**atrix **I**nequality
DAR	**D**ifferential **A**lgebraic **R**epresentation
LFR	**L**inear **F**ractional **R**epresentation
LPV	**L**inear **P**arameter **V**arying
LTI	**L**inear **T**ime **I**nvariant
LDI	**L**inear **D**ifferential **I**nclusion
DOA	**D**omain **O**f **A**ttraction
MIMO	**M**ulti **I**nput **M**ulti **O**utput
SISO	**S**ingle **I**nput **S**ingle **O**utput
SDP	**S**emidefinite **P**rogramming

Notations

$\mathbb{X}$	Polytopic set of real vector $x(t)$
Δ	Polytopic set of real vector δ
χ	Polytopic set of real vector $\Phi(x)$
$\mathbb{Z}$	Polytopic set of real vector $z(t)$
$\mathbb{R}^n$	The set of n-dimentional real vector
$\mathbb{R}^{n \times m}$	The set of $n \times m$ real matrix
$\mathbb{Z} \times \Delta \subset \mathbb{R}^{r+l}$	Cartesian product of $\mathbb{Z}$ and Δ for two sets $\mathbb{Z} \subset \mathbb{R}^r$ and $\Delta \subset \mathbb{R}^l$
$[1, m]$	A set of positive integers from 1 to m
$A_{i_{r_i \times r_i}}$	A real matrix with r_i rows and r_i columns
I_r	An identity matrix with dimension $r \times r$
$diag\{...\}$	A block diagonal matrix
$co\{.\}$	Convex hull of vectors
$\|.\|$	The Euclidean norm of the corresponding real vector
$L_{f_0}^{r_i} h_i$	The r_ith Lie derivative of the scalar function h_i w.r.t. the real vector f_0
$\gamma(.), \alpha(.)$	Class $\mathcal{K}$ functions
$\beta(.,.)$	Class $\mathcal{KL}$ function
$w_1(.), w_2(.), w_3(.)$	Class $\mathcal{K}_\infty$ functions

Chapter 1

Introduction

1.1   Control of Uncertain Nonlinear Dynamical Systems

In the real world, many dynamical systems behave like nonlinear continuous-time systems whose physical parameters are not precisely known. However, sometimes one might be able to determine the bounds of these parameters. Assume the following description of an uncertain nonlinear system whose states time derivatives are described by a nonlinear vector field:

$$\dot{x}(t) = F(x, \delta, u), \tag{1.1}$$

where the state vector $x(t) \in \mathbb{R}^n$, the control vector $u(t) \in \mathbb{R}^m$ and the vector of norm-bounded parametric uncertainties $\delta \in \mathbb{R}^l$.

A very popular and simple method to analyze the stability and/or to design a stabilizing controller for system (1.1) is the linear control approach in the vicinity of a system's equilibrium point. Within this context, the robust stabilization and performance of uncertain *linear* systems was mostly investigated around the decade of 1980 (Leitmann, 1979, Barmish, 1985, Petersen and Hollot, 1986, Schmitendorf, 1988, Madiwale et al., 1989, Khargonekar et al., 1990, Xie and De Souza, 1990, Xie et al., 1992). However, the local linear analysis approach fails to guarantee the stability whenever not only the parametric uncertainties and nonlinearities are both involved but also when one is interested in finding a large enough stabilizing region instead of investigating the stability in a sufficiently small vicinity of an equilibrium point.
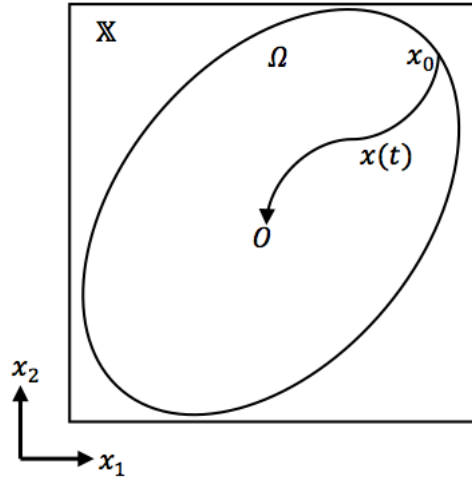
FIGURE 1.1: Stability region Ω in a compact region $\mathbb{X}$ in the state space.

An interesting alternative approach is to look for a compact region, including the equilibrium point, inside which the robust stabilization of uncertain nonlinear systems is guaranteed if the systems states initiate within that region. Figure 1.1 illustrates an example of such a region in a 2-dimensional state space where one of the system's trajectories asymptotically converges to the origin.

According to what was discussed, if we take into account the system's nonlinearities and uncertainties in the robust stability analysis and control synthesis for obtaining a stabilizing region the problem is likely to be less conservative compared to linearization approach for the uncertain nonlinear systems. In this respect, the robust stability analysis pursued in this thesis is based on the notion of polytopic description of set $\mathbb{X} \subset \mathbb{R}^2$ in Figure 1.1 to which the system states belong, and trying to find a guaranteed stabilizing and invariant region $\Omega \subset \mathbb{X}$ in the presence of parametric uncertainties. This region is known as the Domain Of Attraction (DOA) in the literature (see (El Ghaoui and Scorletti, 1996, Tibken, 2000, Hachicho and Tibken, 2002, Chesi, 2004b, Rohr et al., 2009, Coutinho et al., 2009, Chesi, 2009, Zečević and Šiljak, 2010, Ichihara, 2011, Coutinho and De Souza, 2013, Lee, 2013, Trofino and Dezuo, 2014, Gering et al., 2015) and the references therein). In this work we are particularly interested in estimating the largest DOA for a given class of uncertain nonlinear systems by verifying the satisfaction of specific stabilizing Linear Matrix Inequality (LMI) conditions.

1.2   Motivation

The class of uncertain nonlinear systems (1.1) can be recast by different representation and approximation techniques in order to enable LTI control design (see Figure 1.2). Applying these techniques depend on the type of mathematical model of the system. In the context of regional stability and DOA the aforementioned studies investigated different classes of nonlinear systems such as polynomial systems (Tibken, 2000, Hachicho and Tibken, 2002, Chesi, 2004b), non-polynomial systems (Chesi, 2009, Zečević and Šiljak, 2010, Ichihara, 2011), Takagi–Sugeno (T-S) Fuzzy Systems (Lee, 2013, Gering et al., 2015) and rational systems (El Ghaoui and Scorletti, 1996, Rohr et al., 2009, Coutinho et al., 2009, Coutinho and De Souza, 2013, Trofino and Dezuo, 2014). However, when the parametric uncertainties and nonlinearities are taken into account the conservatism of robust stability analysis and control synthesis of the nonlinear systems will rely on how the system is represented. This problem can be addressed while the system states and the uncertainties explicitly show up in the stability analysis instead of being considered as a norm-bounded input perturbation. Accordingly, among the aforementioned classes of nonlinear systems the rational systems, which covers polynomial systems as well, seems to be interesting for investigation since one is able to recast them in Linear Fractional Representation (LFR) and/or Differential Algebraic Representation (DAR) as will be discussed in detail in Chapter 2, Sections 2.2 and 2.3, specifically when the uncertainties are considered. The LFR and DAR representations of rational uncertain nonlinear systems were mostly used in the robust control community, that employs LMIs, as basic tools after it was shown that LMI-based robust stability
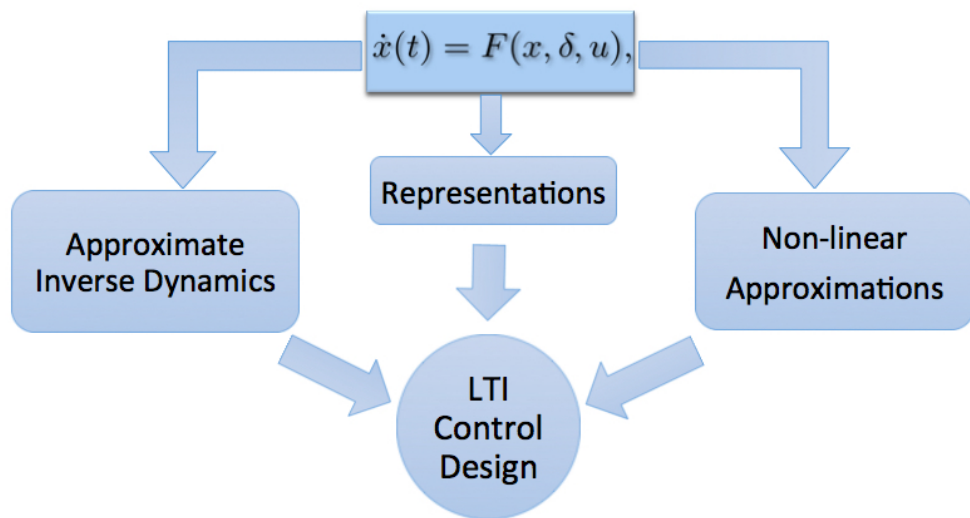
FIGURE 1.2: Different approaches to recast the system.
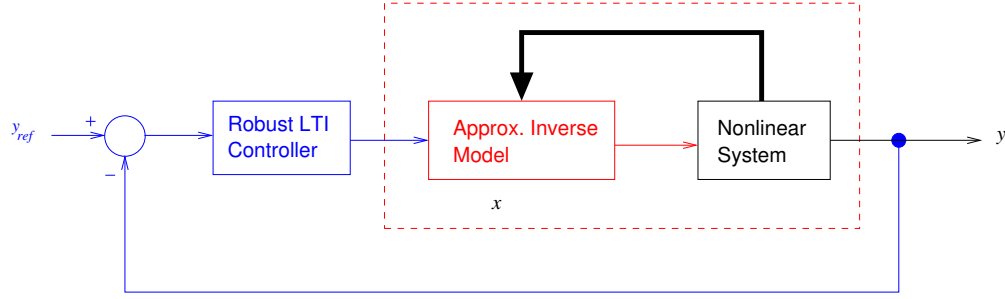
FIGURE 1.3: Inverse Dynamics based Control as a feedback linearization strategy.

and performance analysis can be performed on the corresponding Linear Differential Inclusion (LDI) systems (Boyd et al., 1994). Specifically an LDI system can be reduced to a Linear Time-Invariant (LTI) system for which straightforward systematic robust stability analysis and control synthesis approaches are investigated with different kinds of uncertainties (Khargonekar et al., 1987, Georgiou et al., 1987, Verma, 1989, Huang et al., 2000, Lastman and Sinha, 2001, Cheng and Zhang, 2004, Ebihara and Hagiwara, 2005, Lim et al., 2006, Gonçalves et al., 2006, Lim et al., 2014, Lee et al., 2015).

On the other hand, if system (1.1) is affine in input control we might be able to apply an inverse dynamics method, a nonlinearity cancellation technique, such that an approximate uncertain LTI system around instantaneous operating points is obtained and this provides LTI robust stabilizing control design. The general concept of robust control with inverse dynamics is depicted in block diagram of Figure 1.3. The inverse dynamics method, inside the red dashed box, strongly depends on the exact knowledge of the system structure and parameters (Isidori, 1989, Nijmeijer and Schaft, 1990). The role of the LTI robust controller here is to guarantee stability and performance of the closed-loop system, even when the linearization is not perfect due to the parametric uncertainties, such that the system in the dashed box is only approximately linear and time-invariant. In this context, some studies tackled the inexact linearization problem by proposing different approximately linearizable approaches (Guardabassi, 2004, Deutscher and Schmid, 2006, Jemai et al., 2010, Cardoso and Schnitman, 2011, Menini and Tornambè, 2012, Atam et al., 2014).

A well-known nonlinear control design technique of inverse dynamics is feedback linearization, whose application is intimately related to the idea of canceling nonlinearities aiming to achieve a resulting linear behavior for the system, in new coordinates (Isidori, 1989, Slotine et al., 1991). However, feedback linearization has one significant drawback associated with the

fact that usually one has to assume exact knowledge of the system equations, together with the ideal measurement of all system states (Guardabassi and Savaresi, 2001).

As it will be discussed in Chapter 2, the case of nonlinear systems described by vector fields with uncertain parameters is presented in Section 2.2. Clearly the application of feedback linearization procedures for this class of systems relies on the nominal system model and this leads to another nonlinear, instead of linear, dynamical system in new coordinates as will be explained in Section 2.4. Accordingly, the degree of nonlinearity of the resulting system is actually unknown. In this scenario, a natural approach is to consider the resulting system as a new uncertain nonlinear system, whose dynamics is possibly closer to that exhibited by a genuine LTI system, and for which one has to synthesize robust stabilizing control laws. This idea is interesting because it makes amenable the use of more general synthesis procedures for uncertain nonlinear systems to solve the problem of robust stabilization. Moreover, it becomes specially important in those general methods that seem to arise from extensions of the robust control theory for LTI systems, such as gain scheduling relying on uncertain Linear Parameter Varying (LPV) models (Rotondo et al., 2014), or the use of more detailed representations to describe the nonlinearities in the system dynamics (Wang et al., 1992, El Ghaoui and Scorletti, 1996, Coutinho et al., 2002, Franco et al., 2006, Coutinho et al., 2008, Trofino and Dezuo, 2014).

Among the different approaches that have been reported in the literature to stabilize the resulting nonlinear uncertain system obtained after an attempt of feedback linearization, a key issue seems to be the choice of an appropriate representation for what is left, after such attempt, with respect to what is expected in the absence of uncertainties. This is intimately related to structural properties of the nonlinear part of the system dynamics that are necessary in many methods, e.g. in (Marino and Tomei, 1993). One of these structural properties could be that the remaining uncertain nonlinear part is rational with respect to both parametric uncertainties and states in new coordinates. If such a property is met, as discussed earlier, one can represent the system in LFR and/or DAR forms as it will be shown in Section 2.4.

LFR was studied for nonlinear dynamical systems with vector fields described by rational functions in (El Ghaoui and Scorletti, 1996), where one of the authors contribution was the use of the DOA concept in the analysis of closed-loop regional stability. By generalizing the concept of LFR through the proposition of DAR, which is used to describe not only the nonlinearities, but also the associated parametric uncertainties, Coutinho, Trofino and co-workers (Coutinho
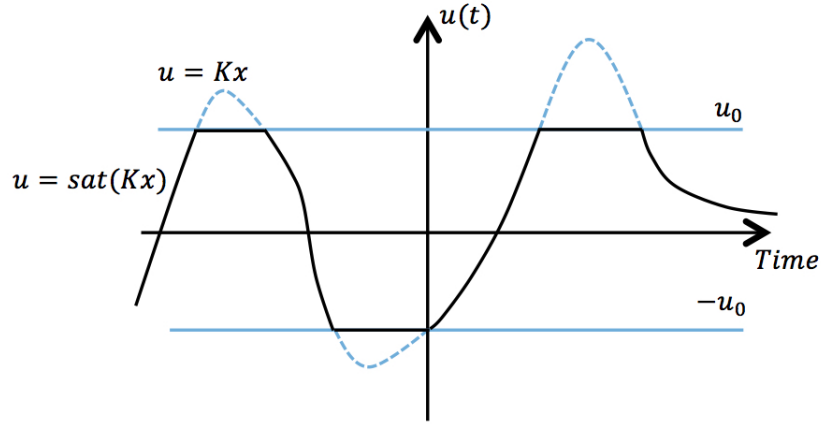
FIGURE 1.4: Saturation of input signal and linear state feedback control command.

et al., 2002, 2008, Trofino and Dezuo, 2014, Coutinho et al., 2009) have obtained very interesting analysis results with respect to the enlargement of DOA, which is either represented as a non-ellipsoidal DOA through polynomial Lyapunov functions or as an ellipsoidal invariant region inside the true DOA of the uncertain nonlinear system (see for example the region Ω in Figure 1.1). The theoretical framework of ellipsoidal DOA will be presented in Section 3.2 of Chapter 3 and it was specifically utilized for the problem of stabilizing feedback linearizable systems in (Rohr et al., 2009). Analogously, for the class of Input-Output feedback linearizable systems with parametric uncertainties, and representable in DAR format, the ellipsoidal DOA in new coordinates can also be addressed by the proposition of sufficient LMIs as it will be discussed in Section 4.1 of Chapter 4. However, previous researches were not restricted to the ellipsoidal DOA of DAR systems such that recent studies aimed to estimate a non-ellipsoidal description of the guaranteed DOA (Chesi, 2004a, Chesi et al., 2004, Coutinho et al., 2008, Coutinho and De Souza, 2013, Coutinho et al., 2009, Trofino and Dezuo, 2014). In this context, they looked for polynomial Lyapunov candidate functions, instead of quadratic ones which characterize ellipsoidal DOA. On the other hand, despite the successful analysis tools, control *synthesis* strategies of DAR systems were only recently presented to estimate the DOA (Oliveira et al., 2012, 2013, Da Silva et al., 2014), in the context of systems with input saturation.

Despite the fact that in practice the inputs of every real system are bounded, sometimes the saturation of input signals can be beneficial in terms of achieving larger guaranteed DOAs and, therefore, better closed-loop stability properties (Hu and Lin, 2001), particularly in the present context of uncertain nonlinear systems controlled by means of static linear state feedback.

As depicted in Figure 1.4, the saturation condition occurs when the linear state feedback control command exceeds the saturation limits. In this respect, many recent works have considered

saturation of inputs, combined with different assumptions and techniques (Barreiro et al., 2002, Castelan et al., 2005, 2008, Coutinho and Da Silva, 2010, Valmórbida et al., 2010, Oliveira et al., 2012, 2013). A particularly interesting approach is the use of the so-called generalized sector condition (Hu et al., 2004) and considering deadzone nonlinearities together with static anti-windup control for a system in DAR form, as recently investigated in (Da Silva et al., 2014). On the other hand, it was shown that the input saturation signal can be recast by the convex combination of the linear state feedback control command and another proper linear vector function whose idea will be presented in Section 2.5 (Hu and Lin, 2001). Within this context, as it will be seen in Section 4.2 of Chapter 4, an interesting approach can be the application of polytopic description of saturated input for the DAR systems in order to evaluate the conjecture that less conservative robust stability analysis and control synthesis problem can be obtained.

1.3 Objectives

In this study we are interested in the broader context of controlling nonlinear dynamical systems, considering parametric uncertainties in their mathematical model representations. Therefore, based on what was discussed above, the following objectives are pursued:

1. First, we attempt to investigate robust control strategy based on inverse dynamics of the system (1.1). For this purpose, in order to apply feedback linearization, we will represent system (1.1) in control-affine form with vector fields having norm-bounded parametric uncertainties and we will consider the class of Input-Output feedback linearizable systems. Owing to the approximate linearization the resulting system in new coordinates is in a quasi-canonical form. Therefore, the DAR representation of such system is handled enabling regional stabilization analysis and control synthesis over the polytopic set of state space and parametric uncertainties. Then, sufficient synthesis LMIs are derived under the condition of Input-to-State Stability (ISS) of the system's internal dynamics.

2. To consider the class of input saturation uncertain nonlinear systems representable in DAR form and to investigate the regional robust stabilization of such systems in terms of DOA estimation we will represent the polytopic description of saturation input as convex combination of unsaturated inputs and appropriate vectors. This polytopic description alongside the DAR representation enables the derivation of synthesis LMIs.

1.4 Summary of Contributions

The first contribution of the present work is the proposition of new sufficient LMI conditions to synthesize robust controllers for input-output feedback linearizable uncertain nonlinear systems, using the DAR approach. The system is assumed to satisfy the property of input-to-state stability of its internal dynamics by providing the corresponding ISS-Lyapunov function. Using this assumption, it will be shown that by solving an SDP problem subject to sufficient LMIs one can estimate a guaranteed ellipsoidal DOA inside the polytopic set of state space in new coordinates.

The second contribution of this work is the proposition of new sufficient LMI conditions to synthesize robust linear state feedback controllers for uncertain Multiple-Input Multiple-Output (MIMO) nonlinear systems with input saturation, assuming that they can be described in a DAR form. A part of the mathematical development relies on the approach proposed in (Hu and Lin, 2001), for linear systems with input saturation, to represent the signals resulting from the saturation of the inputs as convex combinations of unsaturated inputs and appropriate vectors, instead of making direct use of a generalized sector condition (Hu et al., 2004). In addition, the search for the largest *ellipsoidal* guaranteed DOA by means of quadratic Lyapunov candidate functions is pursued to simplify the process of deriving sufficient synthesis conditions. As it will be shown, better results, in the sense of larger guaranteed ellipsoidal DOAs, are obtained when compared to the results recently reported in the literature for the same numerical examples. In this respect, the following paper addresses this contribution:

S. Azizi, L. A. B. Torres & R. M. Palhares (2017): Regional robust stabilisation and domain-of-attraction estimation for MIMO uncertain nonlinear systems with input saturation, International Journal of Control, DOI: 10.1080/00207179.2016.1276634.

1.5 Thesis Organization

The report is written with the following order:
Chapter 2 gives different representation techniques for the uncertain nonlinear systems including LFR and DAR models, representation by applying inverse dynamics and polytopic description of input saturation. Regional stability analysis in the context of ellipsoidal DOA is represented in Chapter 3 where it is shown that how the ellipsoidal DOA is characterized by the quadratic Lyapunov candidate function and how can ensure that the DOA is a subset of the states polytopic

set. Moreover, the condition of input-to-state stability of the internal dynamics for the class of I/O linearizable systems is indicated in this chapter. In Chapter 4 the synthesis problems for the DAR of I/O linearizable system and input saturation system are presented in terms of sufficient LMIs and the estimation of maximum DOAs. Such estimations are computed my means of convex optimization problems subject to the LMIs. Chapter 5 brings some illustrative numerical examples from the literature to examine the effectiveness of the study. To that end, Chapter 6 concludes the thesis with some remarks and possible future research directions.

Chapter 2

Uncertain Nonlinear Dynamical Systems Representations

Representation of the uncertain nonlinear system (1.1) plays an important role in the robust stability analysis and control synthesis. In this respect, there are many research studies that attempted to represent uncertain nonlinear systems both with an admissible approximation and with an exact representation. Some famous representation models are LPV, LFR and DAR forms. In view of obtaining sufficient LMI conditions based on Lyapunov theory, these representations are used to facilitate stability analysis and control synthesis problems. In the rest of this chapter these three representation methods are discussed and compared. Then, we study the problem of trying to represent uncertain nonlinear systems in new coordinates by applying feedback linearization technique aiming nonlinearity reduction in the presence of uncertainties. Finally we justify the importance of investigating uncertain nonlinear systems with input saturation and we represent the saturation input signal as a convex combination of input control command vector and properly chosen vector, such that this combination satisfies saturation condition.

2.1 Linear Parameter Varying Models

As first introduced by Shamma (Shamma, 1988), an LPV representation of system (1.1), in the context of uncertain nonlinear systems, leads to the following state-space formulation:

$$
\begin{aligned}
\dot{x}(t) &= A(\theta(t))x + B(\theta(t))u, \\
y &= C(\theta(t))x + D(\theta(t))u,
\end{aligned}
\tag{2.1}
$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^{n_y}$, $u(t) \in \mathbb{R}^m$, $A(\theta(t)) \in \mathbb{R}^{n \times n}$, $B(\theta(t)) \in \mathbb{R}^{n \times m}$, $C(\theta(t)) \in \mathbb{R}^{n_y \times n}$, $D(\theta(t)) \in \mathbb{R}^{n_y \times m}$, and $\theta(x(t), \delta(t)) \in \Theta \in \mathbb{R}^{n_\theta}$ (Θ is a bounded region) is an exogenous non-stationary vector of parameters that varies inside the region Θ, with the norm-bounded uncertainty $\delta(t)$ belonging to a polytopic set $\Delta \subset \mathbb{R}^l$. The block diagram of this LPV model is depicted in Figure 2.1.

Since then, many studies, mostly published by late 90s, applied this LPV concept to investigate controllers based on the multiplier approach and relying on full block S-procedure (Scherer, 1997, 2001), parametric Lyapunov function based stabilizing LMIs and disturbance attenuation (Kose and Jabbari, 1999, Sato, 2004), gain scheduling output feedback controllers (Sato and Peaucelle, 2013, Hanifzadegan and Nagamune, 2014), robust stability analysis and LPV control design of uncertain polytopic systems (Daafouz et al., 2008, Oliveira and Peres, 2009, Rotondo et al., 2014), fault detection and diagnosis systems (Hecker and Pfifer, 2014). The parameter θ can be an uncertain nonlinear function of system states. For example by considering the following uncertain nonlinear system with time-varying uncertainty $\delta(t)$:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_2 + \frac{\delta(t)(x_1 x_2 + 1)x_2}{1 + \delta(t)} + u,
\end{aligned}
\tag{2.2}
$$
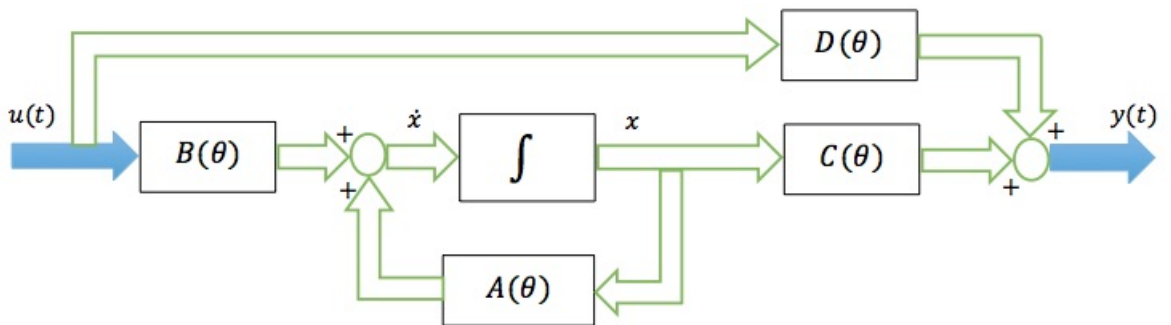
FIGURE 2.1: Block diagram of linear parameter varying model (2.1).

where $|\delta(t)| \leq N$, in which N is the upper bound for $\delta(t)$, we can represent it by the following LPV system:

$$\dot{x}(t) = A(\theta(t))x + Bu, \;\; \theta(t) \in \Theta = \{|\theta(t)| \leq M\}, \tag{2.3}$$

with $\theta(t) = \frac{\delta(t)(x_1 x_2 + 1)}{1 + \delta}$, $A(\theta(t)) = \begin{bmatrix} 0 & 1 \\ 0 & \theta(t) - 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Therefore, the LPV system (2.3) is an approximation of (2.2) for all $x(t)$ and $\delta(t)$ satisfying $|\theta(t)| \leq M$ where the real scalar M characterizes the bounds of the region Θ in (2.3). However, it is not necessary for the two systems (2.2) and (2.3) to be equivalent because one of the main ideas of using LPV representation is to provide the design of linear non-stationary state feedback controllers which are independent from the system states and its nonlinearities. That is, unlike the gain-scheduling approach in which the controllers are dependent to the system nonlinearities, an LPV model consists of an indexed collection of linear systems, in which the indexing parameter is exogenous, i.e., independent of the system states and its nonlinearities.

Based on what is explained above, the advantages of LPV representations over LTI representations of the system's behavior around equilibria is that, first, it is a better system approximation and, second, its feedback control is non-stationary. That is, unlike LTI local representations, the non-stationary feedback control $u = K(\theta(t))x$ in LPV representations can depend on the parameter $\theta(t)$, if $\theta(t)$ is available for measurement or estimation. Also, the LPV model can be built such as it becomes a good approximation of the original system in a region containing equilibrium points. On the other hand, considering the LTI system for stability analysis and control synthesis of the original nonlinear system is only valid in a small neighborhood of the equilibrium point.

2.2 Linear Fractional Representations

As discussed earlier, the choice of an appropriate system representation can facilitate stability analysis and control synthesis. In the one hand approximate representations, such as LPV models, include some nonlinearities of the system encoded as bounded parameters. On the other hand, there exist some representations, such as LFR, which encode uncertainties and nonlinearities by adding more states to the system description and using uncertainty dependent vectors instead of considering them as bounded parameters. Suppose that the uncertain nonlinear system (1.1) belongs to the more specific class of input-affine uncertain nonlinear systems, with the
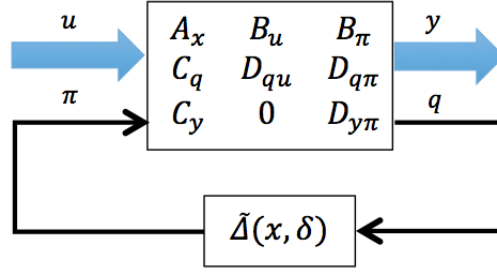
FIGURE 2.2: Linear Fractional Representation of rational system (2.4).

same number of inputs and outputs, given by:

$$
\begin{aligned}
\dot{x}(t) &= f(x,\delta) + \sum_{j=1}^{m} g_j(x,\delta) u_j(t), \\
y_i &= h_i(x), \quad i = 1, 2, ..., m,
\end{aligned}
\tag{2.4}
$$

where x takes values in $\mathbb{X} \subset \mathbb{R}^n$; $u(t) \in \mathbb{R}^m$, with $m \leq n$; $\delta \in \Delta \subset \mathbb{R}^l$ is an uncertain norm-bounded parameter vector which can be divided into nominal and uncertain parts; $f(\cdot) :$ $\mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$, $g_j(\cdot) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ and $h_i(\cdot) : \mathbb{R}^n \to \mathbb{R}$ are smooth vector functions in their arguments, and also are rational in $\mathbb{X} \times \Delta$. Besides, $f(x,\delta)$ satisfies $f(0,\delta) = 0$ for all $\delta \in \Delta$. Here we assumed that there is no uncertainty in the output equations and $h_i(0) = 0$. It should be noted that the control-affine dynamics (2.4), despite being less general than (1.1), still represents a large number of uncertain nonlinear systems. Therefore, the LFR of (2.4) can be obtained as (El Ghaoui and Scorletti, 1996):

$$
\begin{aligned}
\dot{x} &= A_x x + B_u u + B_\pi \pi, \\
q &= C_q x + D_{qu} u + D_{q\pi} \pi, \\
y &= C_y x + D_{y\pi} \pi, \quad \pi = \tilde{\Delta}(x,\delta) q,
\end{aligned}
\tag{2.5}
$$

where $\tilde{\Delta}(x,\delta) = diag\{x_1 I_{r_1}, ..., x_n I_{r_n}, \delta_1 I_{s_1}, ..., \delta_l I_{s_l}\}$ is related to the degree of nonlinearity in the system, $\pi \in \mathbb{R}^{n_\pi}$ is the vector of lumped nonlinearities and uncertainties with $n_\pi = \sum_{i=1}^{n} r_i + \sum_{j=1}^{l} s_j$, in which r_i and s_i are nonnegative integers, and $A_x \in \mathbb{R}^{n \times n}, B_u \in \mathbb{R}^{n \times m}, B_\pi \in \mathbb{R}^{n \times n_\pi}, C_q \in \mathbb{R}^{n_\pi \times n}, D_{qu} \in \mathbb{R}^{n_\pi \times m}, D_{q\pi} \in \mathbb{R}^{n_\pi \times n_\pi}, C_y \in \mathbb{R}^{m \times n}, D_{y\pi} \in \mathbb{R}^{m \times n_\pi}$ are constant matrices. Also the state-space representation (2.5) is such that:

$$
\begin{bmatrix} f(x,\delta) & G(x,\delta) \\ H(x) & 0 \end{bmatrix} = \begin{bmatrix} A_x & B_u \\ C_y & 0 \end{bmatrix} + \begin{bmatrix} B_\pi \\ D_{y\pi} \end{bmatrix} \tilde{\Delta}(x,\delta) \left[I - D_{q\pi} \tilde{\Delta}(x,\delta) \right]^{-1} \begin{bmatrix} C_q & D_{qu} \end{bmatrix},
\tag{2.6}
$$

where $H(x) = [h_1(x), ..., h_m(x)]^T$ and $I - D_{q\pi}\tilde{\Delta}(x, \delta)$ is assumed to be a full rank matrix for well-posed systems. Accordingly, we can associate with the LFR (2.5) an LTI system with some fictitious input π and some fictitious output q as shown in Figure 2.2. This is interesting because, apart from the above definition, for the uncertain state dependent matrix $\tilde{\Delta}(x, \delta)$ it can also be interpreted other different forms of perturbations belonging to a set Λ which describes the size, nature and structure of the uncertainty. In this respect, many works in the literature consider different types of uncertain perturbation matrix $\tilde{\Delta}$ such that the LFR (2.5) can be recast as an LDI system by replacing $\tilde{\Delta}(x, \delta)$ with a time-varying uncertain norm-bounded matrix $\Delta(t)$ (El Ghaoui and Scorletti, 1996, Wang et al., 1992, Chesi et al., 2004, Apkarian and Tuan, 2000, Hentabli et al., 2003, Laroche and Knittel, 2005, De la Sen, 2007, Roos et al., 2010, Chesi, 2010, 2013); or with rational parametric uncertainty belonging to a polytopic set (Cockburn, 1998, Korogui and Geromel, 2009); or with uncertainties due to frequency variations and complex dynamics (Sana and Rao, 2000, Xu et al., 2008, Pfifer and Hecker, 2011, Xu et al., 2012).

As an example of LFR (2.5), system (2.2) can be represented using the following matrices:

$$\pi = \begin{bmatrix} \frac{\delta x_1 x_2^2}{1+\delta} & \frac{\delta x_2^2}{1+\delta} & \frac{\delta x_2}{1+\delta} \end{bmatrix}^T, \tilde{\Delta}(x, \delta) = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & \delta \end{bmatrix},$$

$$A_x = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_\pi = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, C_q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, D_{qu} = 0_{1\times3}, D_{q\pi} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 1 & 0 \end{bmatrix}, D_{y\pi} = 0_{1\times3}.$$

As it can be seen, the LFR of example (2.2) provides constant realization matrices which characterize an LTI system with an uncertain input nonlinear vector comprising the rest of information of the original uncertain nonlinear system. On the contrary, applying LPV representation is likely a conservative approach since we are approximating some system nonlinearities and uncertainties by the time-varying parameter θ which causes missing of information about those nonlinearities.

It is important to note that LFRs of rational systems are not unique and one can obtain different LFRs for the same system. In addition, obtaining LFRs for nonlinear systems, specifically when they have high degrees of nonlinearities, is not a trivial and easy task.

2.3   Differential Algebraic Representations

It was already claimed that the LFR (2.5) corresponds to the rational system (2.4) (El Ghaoui and Scorletti, 1996). A generalization of the idea of representing a system in LFR form is the so-called DAR, which was proposed in (Coutinho et al., 2002, 2008). Since every LFR system can be reformulated as a DAR one, rational systems have also exact, although not unique, DARs (Coutinho et al., 2008, Coutinho and De Souza, 2013). DARs are more general than LFRs in the sense that, instead of having constant matrices, in DARs the matrices can be affine functions of uncertainties and states, such that (2.4) would be rewritten as:

$$\dot{x}(t) = A_1(x,\delta)x + A_2(x,\delta)\pi + A_3(x,\delta)u(t), \tag{2.7}$$

$$0 = \Pi_1(x,\delta)x + \Pi_2(x,\delta)\pi + \Pi_3(x,\delta)u(t),$$

with $\pi \equiv \pi(x,u,\delta) \in \mathbb{R}^{n_\pi}$ any possible and freely chosen vector of nonlinear functions; and $A_1(x,\delta) \in \mathbb{R}^{n \times n}$, $A_2(x,\delta) \in \mathbb{R}^{n \times n_\pi}$, $A_3(x,\delta) \in \mathbb{R}^{n \times m}$, $\Pi_1(x,\delta) \in \mathbb{R}^{n_\pi \times n}$, $\Pi_2(x,\delta) \in \mathbb{R}^{n_\pi \times n_\pi}$ and $\Pi_3(x,\delta) \in \mathbb{R}^{n_\pi \times m}$ are matrices of affine functions with respect to (x,δ), such that $\Pi_2(x,\delta)$ is a square full-rank matrix for all $(x,\delta) \in \mathbb{X} \times \Delta$. From now on, the dependences of A_1, A_2, A_3, Π_1, Π_2, Π_3 on (x,δ) and of π on (x,u,δ) are omitted for clarity of presentation.

To verify the correctness of the DAR (2.7), one can compare (2.4) to the corresponding expression

$$\dot{x}(t) = (A_1 - A_2\Pi_2^{-1}\Pi_1)x + (A_3 - A_2\Pi_2^{-1}\Pi_3)u(t).$$

It is worthy of note that having LFR of system (2.4) one is able to define new matrices $\Pi_1(x,\delta) = \tilde{\Delta}(x,\delta)C_q$, $\Pi_2(x,\delta) = \tilde{\Delta}(x,\delta)D_{q\pi} - I$, $\Pi_3(x,\delta) = \tilde{\Delta}(x,\delta)D_{qu}$, with $\Pi_2(x,\delta)$ being full-rank, such that the following specific DAR is obtained:

$$\dot{x} = A_x x + B_\pi \pi + B_u u, \tag{2.8}$$

$$0 = \Pi_1(x,\delta)x + \Pi_2(x,\delta)\pi + \Pi_3(x,\delta)u,$$

$$y = C_y x + D_{y\pi}\pi,$$

in which the relation of fictitious input/output is replaced by a null equality.

Despite the fact that DAR is more general than LFR obtaining the DAR of uncertain nonlinear system (2.4) is easier than obtaining its LFR because, in DAR (2.7), we are more free to choose nonlinear input π and state and/or uncertainty dependent matrices $A_1 - A_3$, while these matrices

are restricted to be constant in LFR. For an illustration, considering again system (2.2), one possible DAR (2.7) for such a system can be obtained with

$$\pi = \begin{bmatrix} \frac{\delta x_1 x_2}{1+\delta} & \frac{\delta x_2}{1+\delta} \end{bmatrix}^T,$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ x_2 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \Pi_1 = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}, \Pi_2 = \begin{bmatrix} -1 & x_1 \\ 0 & -(\delta+1) \end{bmatrix}, \Pi_3 = 0_{2 \times 1},$$

in which the input nonlinearity dimension is reduced in this DAR representation.

Remark 2.1. It should be noted that DAR representation, as well as LFR, is not unique and one can obtain different DARs for the same system. This fact may lead to different stability analyses, that is, if the DAR of a system is not properly chosen one can cause more conservative stability results.

For DARs in (Coutinho et al., 2002), it was investigated the problem of guaranteed cost control design over LMI conditions by employing polynomial Lyapunov functions. Then, the regional robust stability and performance was investigated in (Coutinho et al., 2008), for the DARs of uncertain nonlinear rational systems, by means of sufficient LMI conditions aiming to analyze input-to-output properties. Similar stability analysis together with the estimation of DOA was performed in (Coutinho and De Souza, 2013) for discrete-time DAR systems. Also the estimation of DOA and robust local stability for a class of implicit polynomial systems was studied in (Coutinho et al., 2009) considering Implicit Bilinear Representation (IBR) which is a representation similar to DAR. Meanwhile, in (Rohr et al., 2009) it was applied dynamic inversion considering a DAR of an uncertain SISO system inside a DOA, and the authors maximized the DOA through a feasibility optimization problem subject to LMI conditions.

A recent study published in (Trofino and Dezuo, 2014) investigated both regional and global robust asymptotic stability of the same type of system using the notion of DAR. They derived LMI conditions using rational Lyapunov functions with respect to states and uncertain parameters. Then, they Maximized the DOA subject to LMIs in the case of regional stability. A more complex Lyapunov function (i.e. non-quadratic one) was used together with the notion of annihilators, and the Finsler lemma in order to achieve less conservative results.

Based on the considerations about the advantages of DAR over LFR and LPV representations we have chosen to use the DAR (2.7) of nonlinear uncertain system (2.4) to investigate regional robust stabilization problems, while estimating the largest ellipsoidal DOA of such systems.

Since it is more straightforward for rational systems to be rewritten in a DAR form, that is the main reason why we are assuming rational vector fields in (2.4). When the mathematical model of the system has some trigonometric functions one can consider, without loss of generality and with no conservativeness, a change of variables which represents trigonometric functions in rational forms. This strategy was used by (Coutinho and Danes, 2006), (Danes and Bellot, 2006) and (Rohr et al., 2009) for robotic and inverted pendulum systems as well. Consider the following change of variable suggested by (Rohr et al., 2009):

$$\theta = 2\arctan(r). \tag{2.9}$$

In this case, one has that

$$\sin(\theta) = \frac{2r}{1+r^2}, \quad \cos(\theta) = \frac{1-r^2}{1+r^2}. \tag{2.10}$$

Note that the domains of θ and r are bounded and the above change of variable does not hold everywhere.

2.4 On the Feedback Linearization of Uncertain Nonlinear Systems

The study of approximate linearization of system (2.4) has been a subject of investigation mostly since 80s (Desoer and Wang, 1980, Krener, 1984, Reboulet and Champetier, 1984, Rugh, 1984). This is interesting because if one could transform the system nonlinearities by a proper nonlinear mapping, the investigation of the robust stabilization problem would become easier. There are many research studies which tackled this issue with different methods of inverse dynamics in order to approximately linearize nonlinear systems with or without uncertainty (Guardabassi and Savaresi, 2001, Deutscher and Schmid, 2006, Jemai et al., 2010, Cardoso and Schnitman, 2011, Menini and Tornambè, 2012, Atam et al., 2014). Looking for an appropriate locally nonlinear mapping a higher than first order approximation by Taylor expansion was investigated in 1984 by Krener (Krener, 1984). At the same time, the problem of pseudo-linearization was investigated by looking for invertible transformations such that the achieved linear model, tangent to the original one, is independent of the operating point (Reboulet and Champetier, 1984). The similar approach of extended linearization for I/O linearization problem was studied by Rugh in 1984 (Rugh, 1984).

Introducing nonlinearity metrics some researchers attempted to calculate and measure the degree of nonlinearity of a system. In this context, they could design a non-exact linearizing controller to achieve approximate linearization of the system. Therefore, different measures of nonlinearity, called nonlinearity indices, were proposed (Desoer and Wang, 1980, Stack and Doyle, 1997). Other authors have investigated systems which are not exactly feedback linearizable, by trying to approximate the original system to a tangent model, which is feedback linearizable, with respect to the equilibrium manifold (Hauser, 1990). A nonlinear H_∞ control theory was applied to the problem of synthesizing approximately I/O-linearizing controllers in (Allgöwer et al., 1994), where the authors have designed a controller that approximately transform a nonlinear system into a linear one by means of minimizing the H_∞ norm associated with model-matching error.

2.4.1 Approximate Feedback Linearization

Feedback linearization is one of the most common techniques of inverse dynamics aiming system's nonlinearity cancellation. The motivation of applying this well-known approach arises from the fact that some control-affine nonlinear systems can be linearized along the instantaneous states by means of a change of coordinates together with an input transformation using state feedback (Isidori, 1989, Nijmeijer and Schaft, 1990). However, feedback linearization has some significant drawbacks. Since it needs the fully knowledge of system and the exact measurement of system states, feedback linearization can fail to stabilize uncertain systems. This is because the inverse of the nonlinear model is not able to completely cancel the real system's nonlinearities due to the existence of uncertainties.

Considering the specific class of input-affine MIMO uncertain nonlinear systems with equal number of inputs and outputs, one can separate system (2.4) into a nominal part and an uncertain part as follows:

$$\dot{x}(t) = f_0(x, p_0) + \sum_{j=1}^{m} g_{0j}(x, p_0) u_j(t) + \Delta f(x, \delta p) + \sum_{j=1}^{m} \Delta g_j(x, \delta p) u_j(t),$$
$$y_i = h_i(x), \quad i = 1, 2, ..., m,$$
(2.11)

where $f = f_0 + \Delta f$, $g_j = g_{0j} + \Delta g_j$ and the parametric uncertainty vector $\delta = p_0 + \delta p$ includes the nominal parameters vector $p_0 \in \mathbb{R}^l$ and the vector of parametric variations δp such that $\delta \in \Delta$ in which the polytopic set $\Delta := \{\delta \in \mathbb{R}^l \mid |\delta p_s| \leq \bar{\delta}_s, \ s = 1, 2, ..., l\}$ where $\bar{\delta}_s$ is the maximum bound of δp_s variations. The I/O feedback linearization of system's nominal part

will be taken in order to reduce the system's nonlinearities. However, due to the existence of uncertainty vector δ, the whole dynamical system cannot be transformed to a canonical form.

To implement I/O feedback linearization consider the following definition (Isidori, 1989, Slotine et al., 1991).

Definition 2.2. The nominal part of system (2.11), i.e. the system (2.11) with $\Delta f(.) \equiv 0$ and $\Delta g_j(.) \equiv 0$, has a (vector) relative degree $\{r_1, ..., r_m\}$ at a point x^o if

$$L_{g_{0_j}} L_{f_0}^k h_i(x) = 0$$

for all $1 \leq j \leq m$, $1 \leq i \leq m$, $0 \leq k < r_i - 1$, and for all x in a neighborhood of x^o, and the $m \times m$ matrix

$$G_* = \begin{bmatrix} L_{g_{01}} L_{f_0}^{r_1-1} h_1 & L_{g_{02}} L_{f_0}^{r_1-1} h_1 & \cdots & L_{g_{0m}} L_{f_0}^{r_1-1} h_1 \\ L_{g_{01}} L_{f_0}^{r_2-1} h_2 & L_{g_{02}} L_{f_0}^{r_2-1} h_2 & \cdots & L_{g_{0m}} L_{f_0}^{r_2-1} h_2 \\ \vdots & \vdots & \ddots & \vdots \\ L_{g_{01}} L_{f_0}^{r_m-1} h_m & L_{g_{02}} L_{f_0}^{r_m-1} h_m & \cdots & L_{g_{0m}} L_{f_0}^{r_m-1} h_m \end{bmatrix},$$

is nonsingular at $x = x^o$. Alternatively, r_i, the associated relative degree of the output channel $h_i(x)$, is the number of times it is required to differentiate $h_i(x)$ until at least one component of the input vector $u(t)$ appears.

Based on the above definition the total relative degree of system (2.4) can be defined as $r = \sum_{i=1}^{m} r_i$, which is not necessarily equal to the system's state vector dimension. Therefore, by considering the following nonlinear mapping:

$$\begin{aligned} \varphi &= \Phi(x) = [z^T \ \zeta^T]^T, \\ z &= [h_1, L_{f_0} h_1, ..., L_{f_0}^{r_1-1} h_1, ..., h_m, ..., L_{f_0}^{r_m-1} h_m]^T, \end{aligned} \tag{2.12}$$

in which Φ takes values inside the set $\chi \subset \mathbb{R}^n$, and differentiating the new state vector φ with respect to time the following will be obtained:

$$
\begin{aligned}
\dot{z}_1 &= \dot{h}_1 = L_f h_1 = L_{f_0} h_1 + L_{\Delta f} h_1 = z_2 + L_{\Delta f} h_1, \\
\dot{z}_2 &= \tfrac{d}{dt}(L_{f_0} h_1) = L_f L_{f_0} h_1 = L_{f_0}^2 h_1 + L_{\Delta f} L_{f_0} h_1 = z_3 + L_{\Delta f} L_{f_0} h_1, \\
&\vdots \\
\dot{z}_{r_1} &= \tfrac{d}{dt}(L_{f_0}^{r_1-1} h_1) = L_f L_{f_0}^{r_1-1} h_1 = L_{f_0}^{r_1} h_1 + \sum_{i=1}^m L_{g_{0i}} L_{f_0}^{r_1-1} h_1 u_i + L_{\Delta f} L_{f_0}^{r_1-1} h_1 \\
&+ \sum_{i=1}^m L_{\Delta g_i} L_{f_0}^{r_1-1} h_1 u_i, \\
&\vdots \\
\dot{z}_{r_m} &= \tfrac{d}{dt}(L_{f_0}^{r_m-1} h_1) = L_f L_{f_0}^{r_m-1} h_1 = L_{f_0}^{r_m} h_1 + \sum_{i=1}^m L_{g_{0i}} L_{f_0}^{r_m-1} h_1 u_i + L_{\Delta f} L_{f_0}^{r_m-1} h_1 \\
&+ \sum_{i=1}^m L_{\Delta g_i} L_{f_0}^{r_m-1} h_1 u_i, \\
\dot{\zeta}(t) &= w(z, \zeta, u, \delta p).
\end{aligned}
\tag{2.13}
$$

Therefore, the approximate I/O feedback linearizing control input which puts system (2.4) in a quasi-canonical form is given by

$$
u = G_*^{-1}(v - f_*),
\tag{2.14}
$$

where

$$
f_* = \left[L_{f_0}^{r_1} h_1, \ L_{f_0}^{r_2} h_2, \ ..., \ L_{f_0}^{r_m} h_m \right]^T,
$$

with v being the new control input vector to be designed. Further, $\zeta \in \mathbb{R}^{n-r}$ is the vector of internal dynamics states. Substituting (2.14) in (2.13), the following system is obtained:

$$
\begin{aligned}
\dot{z}(t) &= A_c z(t) + b_c v(t) + W(z, \zeta, u, \delta p), \\
\dot{\zeta}(t) &= w(z, \zeta, u, \delta p),
\end{aligned}
\tag{2.15}
$$

where $z(t) \in \mathbb{Z} \subset \mathbb{R}^r$, and $\mathbb{Z} \subset \chi$, $v(t) = [v_1(t), v_2(t), ..., v_m(t)]^T \in \mathbb{R}^m$ and $A_c = diag\{A_1, A_2, ..., A_m\}$ in which

$$
A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{r_i \times r_i},
$$

$b_c = \begin{bmatrix} b_1^T & b_2^T & \cdots & b_m^T \end{bmatrix}^T$, with each $b_j \in \mathbb{R}^{r_j}$, $j = 1, 2, \ldots, m$, a r_j-dimensional vector of $r_j - 1$ zeros appended by the last element equal to one; and $W(z, \zeta, u, \delta p) \equiv W(z, \zeta, v, \delta p)$ is the vector in which all remaining nonlinearities associated with the inexact I/O linearization are grouped together, such that

$$W(z, \zeta, u, \delta p) = \begin{bmatrix} L_{\Delta f} h_1 + \sum_{j=1}^{m} L_{\Delta g_j} h_1 u_j \\ L_{\Delta f} L_{f_0} h_1 + \sum_{j=1}^{m} L_{\Delta g_j} L_{f_0} h_1 u_j \\ \vdots \\ L_{\Delta f} h_m + \sum_{j=1}^{m} L_{\Delta g_j} h_m u_j \\ \vdots \\ L_{\Delta f} L_{f_0}^{r_m-1} h_m + \sum_{j=1}^{m} L_{\Delta g_j} L_{f_0}^{r_m-1} h_m u_j \end{bmatrix},$$

with $u = [u_1, u_2, \ldots, u_m]^T$ given by (2.14). Therefore, due to parametric uncertainties, the nonlinear control (2.14) can only linearize the nominal part of system (2.11) and, as a result, the system in φ-coordinates is approximately transformed into a normal form (Isidori, 1989).

The uncertainty vector $W(z, \zeta, u, \delta p)$, in quasi-canonical system (2.15), should be taken into consideration in order to accomplish robust control of the system. One possible approach is to consider that $W(z, \zeta, u, \delta p) \equiv W(t)$ is just a norm bounded external disturbance signal (Marino and Tomei, 1993, Joo and Seo, 1996, Jong-Tae, 2000, 2004, Shan et al., 2007); but this assumption amounts to loss of information about the internal structure of $W(z, \zeta, u, \delta p)$, together with the not easily justifiable fact that this signal is considered *a priori* bounded. Within this context, what makes this work distinct from previous studies is the exact representation of I/O quasi-canonical system explicitly in terms of new state vector φ and uncertainties such that the structural information of $W(z, \zeta, u, \delta p)$ is taken into account in stability analysis.

Whereas the original uncertain nonlinear system (2.4) is comprised by rational vector functions, the I/O linearization process leads to a set of rational systems (2.15) in new coordinates. Therefore, as depicted in Figure 2.3, one can represent the external part of (2.15), which is in quasi-canonical form, into the DAR system such that it enables regional robust stabilization analysis and control synthesis. Therefore, the quasi-canonical system (2.15) can be recast as the

following DAR system in (z, ζ)-coordinates:

$$\dot{z}(t) = A_1(z, \zeta, \delta p)z + A_2(z, \zeta, \delta p)\pi + A_3(z, \zeta, \delta p)v, \qquad (2.16)$$

$$0 = \Pi_1(z, \zeta, \delta p)z + \Pi_2(z, \zeta, \delta p)\pi + \Pi_3(z, \zeta, \delta p)v.$$

Now, one can investigate the regional robust stabilization of system (2.16) by synthesizing a linear feedback control for the input v as it will be described in Chapter 4. We will show later that if the system has input-to-state stable internal dynamics one can prove the closed-loop system stability.

Regarding robustifying techniques applied to feedback linearizable systems, some literature are reviewed here to have a comparison with our approach. In (Marino and Tomei, 1993) a robust global stabilizing state feedback control is designed for a time varying single-input nonlinear system whose parameters uncertainties are unmodelled but bounded. The feedback linearized system disturbed by unknown nonlinearities is shown to be globally stabilizable by a fixed dynamic state feedback compensator while the time varying nonlinear uncertainties satisfy a structural coordinate-free triangularity condition. They improved the stabilization results in (Kanel-lakopoulos et al., 1991) by removing the linear parametrization assumption and considering the bounded time-varying parameters and uncertainties on nonlinearities. In (Joo and Seo, 1996) it was investigated the robust stability of a SISO feedback linearized system containing parametric uncertainties and input disturbance, and it is shown that a linear parametrized transformed system can be derived from an uncertain nonlinear system with single and affine input control.
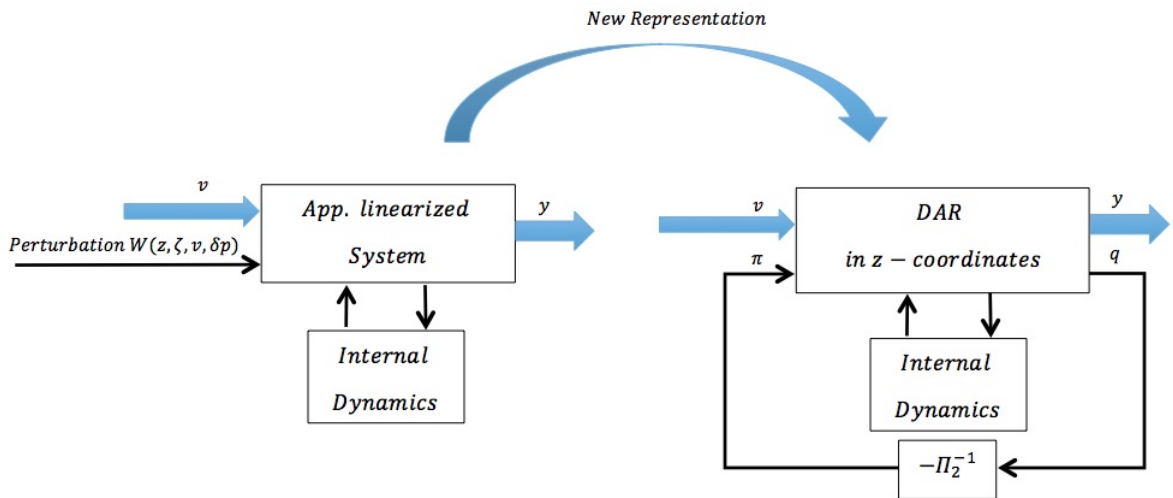
FIGURE 2.3: Representation of I/O linearized model to DAR model.

The resulting parameter varying system, which is strongly accessible from any initial condition, is proved to be globally stabilizable by static-state feedback control law.

An exact feedback linearizable model under unstructured uncertainty is considered in (Chao et al., 1994). The stability robustness is guaranteed using two conditions, the relation between L_2 induced norm and the Hamilton-Jacobi inequality for the nonlinear system, and also the relation between L_2 induced norm of the nonlinear system and H_∞ norm of its linearization. The robust stabilization of uncertain input-state feedback linearizable system, and approximate feedback linearization are also investigated in (Jong-Tae, 2000), (Jong-Tae, 2004), (Guardabassi and Savaresi, 2001) and (Shan et al., 2007). Meanwhile, a class of time-varying nonlinear systems was taken under consideration by (Han-Lim and Jong-Tae, 2003). They developed a stabilizing feedback linearization control scheme by proposing the concept of a time varying diffeomorphism. By representing the uncertain disturbed nonlinear system with Tagaki Sugeno (*TS*) fuzzy model, an LMI-based L_2 robust stability synthesis was performed on fuzzy feedback linearized system by (Park et al., 2004).

The new concept of robust feedback linearization, in which the Brunovsky canonical form of linearized system differs form that of classical feedback linearization, was first introduced in (Guillard and Bourlès, 2000). It is done by performing a clever transformation in the original change of coordinates (or diffeomorphism) and in the original linearizing control law of the nonlinear system. However, this technique works only in a small neighborhood of some operating points.

In (Driemeyer Franco et al., 2005) and (Franco et al., 2006) it was applied robust feedback linearization technique together with a robust linear H_∞ controller for uncertain magnetic bearing system. In (Mokhtari et al., 2005) mixed robust feedback linearization with GH_∞ controller was also used for the application of nonlinear quad-rotor UAV. A MIMO twin rotor application using Lyapunov based robust feedback linearization scheme was investigated as well in (Karimi and Jahed Motlagh, 2006). And the research in (Liu and Soffker, 2009) investigated the disturbance rejection of I/O linearizable system by applying feedback of the system uncertainties and modeling error which are estimated by a specific high-gain PI observer and by states measurements.

Overall, one major challenge of the reviewed studies above is that how can deal with the uncertain nonlinearities of feedback linearizable systems (corresponding to the vector $W(z, \zeta, u, \delta p)$

in system (2.15)). In this context, some assumptions, such as structural coordinate-free triangularity, and some strategies, such as estimation and approximation methods were utilized for those nonlinearities. In the current study, however, we utilize the instrumental DAR tool to have an exact representation of uncertain nonlinearities such that we can incorporate their structural information in the robust stability analysis and control synthesis.

2.4.1.1   Approximate Feedback Linearization of Inverted Pendulum

This section presents an inverted pendulum system model in order to apply approximate feedback linearization strategy. The same inverted pendulum model studied in (Rohr et al., 2009) is used here. The differential equation of this system is

$$\ddot{\theta}(t) = \frac{g}{l}\sin(\theta(t)) - \frac{b\dot{\theta}(t)}{M} + \frac{u(t)}{Ml^2}, \tag{2.17}$$

where g is the gravitational acceleration, l is the length of pendulum, M is the total mas and b is the damping coefficient. Also the state $\theta(t)$ is the angle of pendulum bar with respect to the vertical direction, and $u(t)$ is the control torque actuation. Figure 2.4 depicts the schematic of inverted pendulum dynamics. Note that for simplicity the explicit dependencies on time for states and control signals will not be shown in the next expressions.

According to the change of variables in (2.9) and (2.10) the rational system identical to (2.17) is

$$\ddot{r} = \frac{2r\dot{r}^2}{1+r^2} - \frac{b\dot{r}}{M} + \frac{gr}{l} + \frac{u(1+r^2)}{2Ml^2}. \tag{2.18}$$
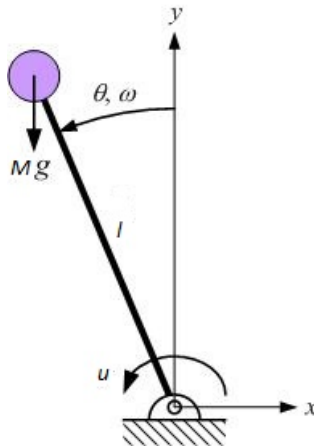
FIGURE 2.4: Schematic of the inverted pendulum dynamics

To show the system (2.18) in state-space form we define $x_1 = r$ and $x_2 = \dot{r}$, such that

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{2x_1 x_2^2}{1+x_1^2} + \frac{g}{l}x_1 - \frac{b}{M}x_2 + \frac{1+x_1^2}{2Ml^2}u, \\
y &= x_1.
\end{aligned}
\tag{2.19}
$$

Suppose that there exist uncertainties in the parameters b and M. The nominal and uncertain parts of them can be written together as $b_0(1 + \delta_2)$ and $M_0(1 + \delta_1)$ in which the uncertainty vector is $\delta p = [\delta_1 \ \delta_2]^T$. Therefore, the uncertain nonlinear rational system is

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{2x_1 x_2^2}{1+x_1^2} + \frac{g}{l}x_1 - \frac{b_0(1+\delta_2)}{M_0(1+\delta_1)}x_2 + \frac{1+x_1^2}{2M_0(1+\delta_1)l^2}u, \\
y &= x_1.
\end{aligned}
\tag{2.20}
$$

Rewriting system (2.20) in the form (2.4) the following drift and steering vector fields, respectively, are obtained:

$$
f(x,\delta) = \begin{bmatrix} x_2 \\ \frac{2x_1 x_2^2}{1+x_1^2} + \frac{g}{l}x_1 - \frac{b_0(1+\delta_2)}{M_0(1+\delta_1)}x_2 \end{bmatrix}, g(x,\delta) = \begin{bmatrix} 0 \\ \frac{1+x_1^2}{2M_0(1+\delta_1)l^2} \end{bmatrix},
$$

with $x = [x_1 \ x_2]^T$. Hence, (2.20) can be rewritten as (2.11) by taking

$$
f_0(x,p_0) = \begin{bmatrix} x_2 \\ \frac{2x_1 x_2^2}{1+x_1^2} + \frac{g}{l}x_1 - \frac{b_0}{M_0}x_2 \end{bmatrix}, g_0(x,p_0) = \begin{bmatrix} 0 \\ \frac{1+x_1^2}{2M_0l^2} \end{bmatrix},
$$

$$
\Delta f(x,\delta p) = \begin{bmatrix} 0 \\ -\frac{b_0(\delta_2-\delta_1)}{M_0(1+\delta_1)}x_2 \end{bmatrix}, \Delta g(x,\delta p) = \begin{bmatrix} 0 \\ -\frac{(1+x_1^2)\delta_1}{2M_0l^2(1+\delta_1)} \end{bmatrix}.
$$

By differentiating twice the output y the input u appears, which means the system has full relative degree ($r = n = 2$) and does not have internal dynamics. Then, by considering the mapping $z = [x_1 \ x_2]^T$ one can obtain the following linearizing control from (2.14)

$$
u(t) = (L_{g_{01}}L_{f_0}x_1)^{-1}\left(v - L_{f_0}^2 x_1\right),
\tag{2.21}
$$

where $L_{g_{01}}L_{f_0}x_1 = \frac{1+x_1^2}{2M_0l^2}$ and $L_{f_0}^2 x_1 = \frac{2x_1 x_2^2}{1+x_1^2} + \frac{g}{l}x_1 - \frac{b_0}{M_0}x_2$. Therefore, the feedback linearized system (2.15) is achieved in which

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$W(z, u, \delta p) = \begin{bmatrix} L_{\Delta f} x_1 \\ L_{\Delta f} L_{f_0} x_1 + L_{\Delta g} L_{f_0} x_1 u \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{b_0(\delta_2 - \delta_1)}{M_0(1+\delta_1)} z_2 - \frac{(1+z_1^2)\delta_1}{2M_0 l^2(1+\delta_1)} u \end{bmatrix}.$$

According to (2.21) we substitute u with v in vector W. So the approximately feedback lin-earized system is

$$\dot{z} = A_c z + b_c v + \begin{bmatrix} 0 \\ -\frac{b_0 \delta_2}{M_0(1+\delta_1)} z_2 + \frac{\delta_1}{1+\delta_1}\left(\frac{2z_1 z_2^2}{1+z_1^2} + \frac{g}{l} z_1\right) - \frac{\delta_1}{1+\delta_1} v \end{bmatrix}. \qquad (2.22)$$

2.4.1.2 Approximate Input/Output Linearization of a MIMO System

The following MIMO system, having internal dynamics, is taken from (Zhang and Bien, 2000):

$$\begin{aligned}
\dot{x_1} &= x_4 - (1 + \delta_2)x_1, \\
\dot{x_2} &= x_2 + x_1 x_3(1 + \delta_1) + (1 + 2(1 + \delta_1)x_3)u_1 + u_2, \\
\dot{x_3} &= x_1^2 - x_2 - (1 + \delta_1)x_3 + 2(1 + \delta_1)x_3 u_1 + u_2, \qquad (2.23) \\
\dot{x_4} &= (1 + \delta_1)x_3, \\
y &= h(x) = \begin{bmatrix} x_2 - x_3 & x_4 \end{bmatrix}^T,
\end{aligned}$$

in which $\delta p = [\delta_1 \ \delta_2]^T$.

Rewriting (2.23) in the form (2.11), with $x = [x_1 \ x_2 \ x_3 \ x_4]^T$ and the control vector $u = [u_1 \ u_2]^T$, we have

$$f_0(x, p_0) = \begin{bmatrix} x_4 - x_1 \\ x_2 + x_1 x_3 \\ x_1^2 - x_2 - x_3 \\ x_3 \end{bmatrix}, g_0(x, p_0) = \begin{bmatrix} 0 & 0 \\ 1 + 2x_3 & 1 \\ 2x_3 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\Delta f(x, \delta p) = \begin{bmatrix} -\delta_2 x_1 \\ \delta_1 x_1 x_3 \\ -\delta_1 x_3 \\ \delta_1 x_3 \end{bmatrix}, \Delta g(x, \delta p) = \begin{bmatrix} 0 & 0 \\ 2\delta_1 x_3 & 0 \\ 2\delta_1 x_3 & 0 \\ 0 & 0 \end{bmatrix}.$$

Differentiating once the output $y_1 = h_1(x) = x_2 - x_3$ and twice $y_2 = h_2(x) = x_4$ the elements of input vector u appear, which means that the total relative degree is $r = 3$ and there exists internal dynamics since $r < 4$. In this respect, the mapping $\Phi(x) = [z^T \ \zeta]^T = [h_1 \ h_2 \ L_{f_0} h_2 \ \zeta]^T$, in which $L_{f_0} h_2 = x_3$ and $\zeta = x_1$, can be considered in order to obtain the linearizing control input from (2.14) as:

$$u = \begin{bmatrix} L_{g_{01}} h_1 & L_{g_{02}} h_1 \\ L_{g_{01}} L_{f_0} h_2 & L_{g_{02}} L_{f_0} h_2 \end{bmatrix}^{-1} \left(v - \begin{bmatrix} L_{f_0} h_1 \\ L_{f_0}^2 h_2 \end{bmatrix} \right), \tag{2.24}$$

where

$$\begin{bmatrix} L_{g_{01}} h_1 & L_{g_{02}} h_1 \\ L_{g_{01}} L_{f_0} h_2 & L_{g_{02}} L_{f_0} h_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2x_3 & 1 \end{bmatrix}, \begin{bmatrix} L_{f_0} h_1 \\ L_{f_0}^2 h_2 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_3 - x_1^2 + x_1 x_3 \\ x_1^2 - x_2 - x_3 \end{bmatrix},$$

and $v \in \mathbb{R}^2$ is the new control input. Therefore, the approximately feedback linearized system for (2.23) is obtained as

$$\dot{z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} v + \delta_1 z_3 \begin{bmatrix} 1 + \zeta \\ 1 \\ -2\zeta z_3 + 2\zeta^2 - 6z_3 - 4z_1 - 1 + 2v_1 \end{bmatrix}, \tag{2.25}$$

and the corresponding internal dynamics is

$$\dot{\zeta} = z_2 - (1 + \delta_2)\zeta. \tag{2.26}$$

2.5 On the Representation of Input Saturation

As earlier discussed in Section 1.2 whereas every dynamical system in practice is subject to some actuation limit, by considering this characteristic in the system's nonlinear model sometimes it is expected to achieve better stability results, which means, in the present context, DOA
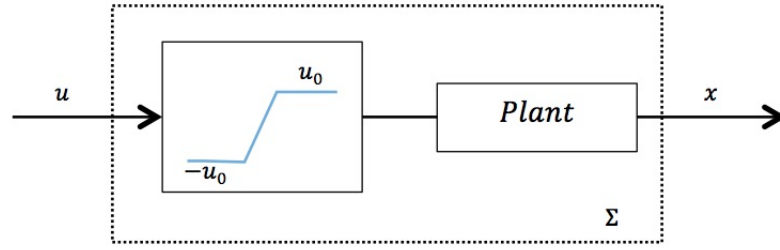
FIGURE 2.5: Input-saturated uncertain nonlinear system.

estimates with higher volume. On the other hand, ignoring the saturation condition in the system model and control design might, in extreme case, lead to instability in practical situations in which actuator saturates while higher control command is demanded (Hu and Lin, 2001). Even though one can neglect saturation in system model and prevent control command from exceeding actuator limits, this strategy is conservative and may cause lower stability performance compared to when saturation is taken into account.

In this thesis we will also investigate regional robust stabilization and DOA estimation, by designing a linear state feedback control with saturated input as shown in Figure 1.4, for the MIMO uncertain input-affine system (2.4) with only one difference that the input control $u(t)$ is replaced by its corresponding saturation vector function $\text{sat}(u(t))$. Therefore, system (2.4) is rewritten as:

$$\Sigma: \ \dot{x}(t) = f(x, \delta) + \sum_{j=1}^{m} g_j(x, \delta)\text{sat}(u_j(t)), \tag{2.27}$$

where saturation input element $\text{sat}(u_j(t))$ is defined as

$$\text{sat}(u_j(t)) := \text{sign}(u_j(t)) \times \min\{|u_j(t)|, u_{0_j}\}; \qquad j = 1, ..., m; \tag{2.28}$$

where $u_{0_j} \in \mathbb{R}^+$ is the maximum absolute value of $u_j(t)$. The typical view of system (2.27) is shown in Figure 2.5.

Interestingly, we can also represent the class of input-saturated systems (2.27) in the following DAR format:

$$\dot{x}(t) = A_1(x, \delta p)x + A_2(x, \delta p)\pi + A_3(x, \delta p)\text{sat}(u(t)), \tag{2.29}$$

$$0 = \Pi_1(x, \delta p)x + \Pi_2(x, \delta p)\pi + \Pi_3(x, \delta p)\text{sat}(u(t)),$$

with $\delta = p_0 + \delta p$, $\pi \equiv \pi(x, \text{sat}(u), \delta p) \in \mathbb{R}^{n_\pi}$ and $A_1 \in \mathbb{R}^{n \times n}$, $A_2 \in \mathbb{R}^{n \times n_\pi}$, $A_3 \in \mathbb{R}^{n \times m}$,
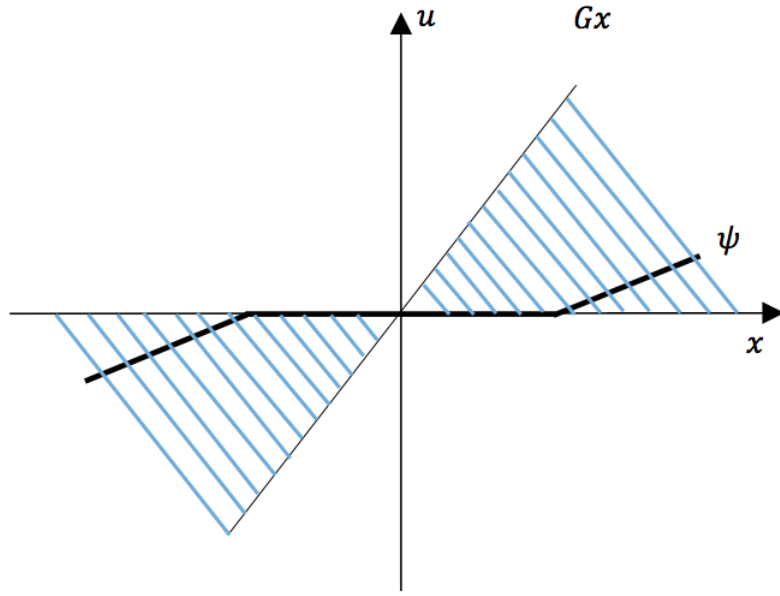
FIGURE 2.6: Dead-zone nonlinearity and sector bound condition.

$\Pi_1 \in \mathbb{R}^{n_\pi \times n}$, $\Pi_2 \in \mathbb{R}^{n_\pi \times n_\pi}$ and $\Pi_3 \in \mathbb{R}^{n_\pi \times m}$ being matrices of affine functions with respect to $(x, \delta p)$, such that Π_2 is a square full-rank matrix for all $(x, \delta) \in \mathbb{X} \times \Delta$.

2.5.1 Description of Input Saturation

The input saturation $\mathrm{sat}(u)$ can be described in different ways for the sake of facilitating stability analysis. One way is to define the so-called dead-zone nonlinearity vector as

$$\psi(t) := u(t) - \mathrm{sat}(u(t)), \ \psi(t) \in \mathbb{R}^m, \tag{2.30}$$

in which $\psi(t)$ takes part as a new input to the system (2.27) such that

$$\dot{x}(t) = f(x, \delta) + \sum_{j=1}^{m} g_j(x, \delta) u(t) - \sum_{j=1}^{m} g_j(x, \delta) \psi(t). \tag{2.31}$$

Note that the above system can also be represented in a DAR model. If one considers the linear state feedback control design $u(t) = Kx(t)$, with $K \in \mathbb{R}^{m \times n}$, and also a real matrix $G \in \mathbb{R}^{m \times n}$ such that

$$- u_0 \le (K - G)x \le u_0, \tag{2.32}$$

$u_0 = [u_{01} \, u_{02} \cdots u_{0m}]^T$, and considering the inequality above element-wise, then the following matrix inequality condition can be inferred (Da Silva and Tarbouriech, 2005):

$$\psi(t)^T M[\psi(t) - Gx] \leq 0, \ M \in \mathbb{R}^{m \times m}, \ M > 0, \tag{2.33}$$

where M is diagonal and positive definite with $M_{(jj)}$ being its jth diagonal element, and this inequality is referred to as the sector bound condition. Figure 2.6 shows the dead-zone non-linearity $\psi(t)$ for the linear state feedback control together with the sector bound condition in shaded space. To verify sector condition (2.33) three cases may happen based on (2.28):

1. $\text{sat}(u_j(t)) = u_j(t)$: Then $\psi_j(t) = 0$ and $\psi_j(t) M_{(j,j)}[\psi_j(t) - G_j x] = 0$, where $\psi_j(t)$ is the jth element of vector $\psi(t)$ and G_j is the jth row of matrix G.

2. $\text{sat}(u_j(t)) = u_{0_j}$ $\left(\text{it happens when } K_j x \geq u_{0_j}\right)$: Then $\psi_j(t) = K_j x - u_{0_j} \geq 0$ with K_j being the jth row of K and, from (2.32), $K_j x - G_j x \leq u_{0_j}$ which means $\psi_j(t) - G_j x \leq 0$. Then, one can conclude that $\psi_j(t) M_{(j,j)}[\psi_j(t) - G_j x] \leq 0$.

3. $\text{sat}(u_j(t)) = -u_{0_j}$ $\left(\text{it happens when } K_j x \leq -u_{0_j}\right)$: Then $\psi_j(t) = K_j x + u_{0_j} \leq 0$ and, from (2.32), $-u_{0_j} \leq K_j x - G_j x$ which means $0 \leq \psi_j(t) - G_j x$. Then, one can conclude that $\psi_j(t) M_{(j,j)}[\psi_j(t) - G_j x] \leq 0$.

Since for all $j \in [1, m]$ the inequality $\psi_j(t) M_{(j,j)}[\psi_j(t) - G_j x] \leq 0$ is true one can infer

$$\sum_{j=1}^{m} \psi_j(t) M_{(j,j)}[\psi_j(t) - G_j x] \leq 0,$$

which results in (2.33). The sector bound condition (2.33) can be incorporated with Lyapunov theory to develop stabilizing LMIs for system (2.29) or (2.27) and this approach has been used broadly to study the systems with input saturation in (Castelan et al., 2006, Garcia et al., 2007, Da Silva et al., 2008, Flores et al., 2009, Garcia et al., 2009, Da Silva et al., 2009, Flores et al., 2010, Coutinho and Da Silva, 2010, Oliveira et al., 2010, Bender et al., 2011, Flores et al., 2012, Da Silva and Turner, 2012, Oliveira et al., 2012, Flores et al., 2013, Da Silva et al., 2013, Oliveira et al., 2013, Da Silva et al., 2014).

On the other hand there is another clever representation of saturation input which is based on the polytopic description of the nonlinear vector $\text{sat}(u(t))$. Following the work in (Hu and Lin, 2001), the saturation vector function $\text{sat}(u(t))$ belongs to the convex hull of a set of two vectors, as stated in the following Lemma adapted from Lemma 7.3.2 of (Hu and Lin, 2001):

Lemma 2.3. *Define*

$$\mathcal{D} := \{D_s : s = 1, 2, ..., 2^m\},$$

to be a set of $m \times m$ diagonal matrices D_s whose diagonal elements are either 0 or 1. Clearly, the number of members of the set $\mathcal{D}$ is 2^m. Also, denoting $D_s^- = I_m - D_s$, then $D_s^- \in \mathcal{D}$. There exists a vector $\tilde{u}(t) \in \mathbb{R}^m$, whose components satisfy $|\tilde{u}_j(t)| \leq u_{0_j}, \forall j \in [1, m]$, such that

$$\text{sat}(u(t)) \in co\{D_s u(t) + D_s^- \tilde{u}(t) : s = 1, 2, ..., 2^m\} \tag{2.34}$$

Proof. Let $u_j := u_j^{(1)}$ and $\tilde{u}_j := u_j^{(2)}$. For only one input vector, i.e. $m = 1$ and $j = 1$ we verify from $|u_1^{(2)}| \leq u_{0_1}$ that:

- $\text{sat}(u_1^{(1)}) = u_1^{(1)} \in co\{u_1^{(1)}, u_1^{(2)}\}$,

- $\text{sat}(u_1^{(1)}) = u_{0_1},\ u_1^{(2)} \leq u_{0_1} \leq u_1^{(1)} \implies \text{sat}(u_1^{(1)}) \in co\{u_1^{(1)}, u_1^{(2)}\}$,

- $\text{sat}(u_1^{(1)}) = -u_{0_1},\ u_1^{(1)} \leq -u_{0_1} \leq u_1^{(2)} \implies \text{sat}(u_1^{(1)}) \in co\{u_1^{(1)}, u_1^{(2)}\}$.

For $m = 2$ ($j \in [1, 2]$) we can infer from the proof for $m = 1$ that there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\sum_{i=1}^2 \alpha_i = \sum_{k=1}^2 \beta_k = 1$ and $\text{sat}(u_1^{(1)}) = \sum_{i=1}^2 \alpha_i u_1^{(i)}$ and $\text{sat}(u_2^{(1)}) = \sum_{k=1}^2 \beta_k u_2^{(k)}$. Then

$$\text{sat}\left(\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix}\right) = \begin{bmatrix} \sum_{i=1}^2 \alpha_i u_1^{(i)} \\ \sum_{k=1}^2 \beta_k u_2^{(k)} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^2 \alpha_i u_1^{(i)} \left(\sum_{k=1}^2 \beta_k\right) \\ \sum_{k=1}^2 \beta_k u_2^{(k)} \left(\sum_{i=1}^2 \alpha_i\right) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^2 \sum_{k=1}^2 \alpha_i \beta_k u_1^{(i)} \\ \sum_{i=1}^2 \sum_{k=1}^2 \alpha_i \beta_k u_2^{(k)} \end{bmatrix}$$

$$= \sum_{i=1}^2 \sum_{k=1}^2 \alpha_i \beta_k \begin{bmatrix} u_1^{(i)} \\ u_2^{(k)} \end{bmatrix} = \sum_{i=1}^2 \alpha_i \sum_{k=1}^2 \beta_k \begin{bmatrix} u_1^{(i)} \\ u_2^{(k)} \end{bmatrix} \in co\left\{ \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix}, \begin{bmatrix} u_1^{(1)} \\ u_2^{(2)} \end{bmatrix}, \begin{bmatrix} u_1^{(2)} \\ u_2^{(1)} \end{bmatrix}, \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} \right\}$$

$$= co\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \right\}.$$

And finally by induction one can continue the proof up to the arbitrary number m of inputs ($j \in [1, m]$) such that (2.34) is obtained. □

If we assume that the input control command $u(t)$ is computed as an LTI feedback of states, i.e. $u = Kx$, and also the vector $\tilde{u}(t)$ in (2.34) is a linear vector of system states, for example $\tilde{u}(t) = Hx$, then, the saturation input can be represented by the convex combination of these linear vectors as depicted in Figure 2.7.
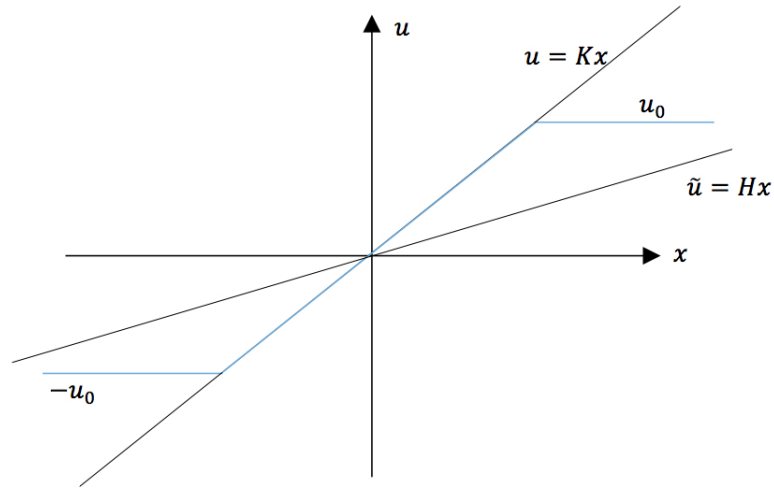
FIGURE 2.7: Representation of saturated input by the convex combination of linear vectors.

Remark 2.4. In light of Lemma 2.3, $\text{sat}(u(t))$ can be replaced in (2.29) by some linear combination of the vectors $u(t)$ and $\tilde{u}(t)$. Therefore, the DAR (2.29) of input-saturated system (2.27) gives us a powerful tool to generalize the polytopic description for input-saturated *nonlinear* systems using the same approach employed in (Hu and Lin, 2001, Hu et al., 2004) for input-saturated *linear* systems. In the other words, the saturated input vector $\text{sat}(u(t))$ is formed by the convex combination of some elements from $u(t)$ and the rest from $\tilde{u}(t)$. This property can be directly applied in the derivation of Lyapunov-based stabilizing LMIs, and is also applied in other works for input-saturated *linear* systems (Hu et al., 2002, Hu and Lin, 2003a, Cao et al., 2002, Fang et al., 2004, Hu et al., 2005, Hu and Lin, 2003b). However, in the context of deriving stabilizing LMIs, it is important to emphasize that this approach can be used at the expense of having 2^m inequalities to be considered.

Overall, the use of property (2.34), which represents the input nonlinearity $\text{sat}(u(t))$ in a linear form, gives us an advantage over dead-zone nonlinearity and sector bound condition. That is, the use of sector bound condition is mostly applicable with S-procedure Lemma, which will be stated in Chapter 4, that induces conservativeness to the stabilizing LMI conditions. On the contrary, the use of property (2.34) for deriving LMI conditions is independent of S-procedure. We will show in Chapter 4 that how one can synthesize linear feedback control of saturated input by representing it with a convex combination of two linear vectors based on (2.34).

Chapter 3

Stability Analysis of Uncertain Nonlinear Dynamical Systems

In this chapter we will use the concepts of regional stability throughout the definition of DOA for the set of systems (2.15), which was the transformation of (2.4), and analogously for the input saturated system (2.27). These concepts will be investigated within the context of Lyapunov quadratic stability which corresponds to obtaining the guaranteed *ellipsoidal* DOA of such systems. Moreover, we will recall the input-to-state stability (ISS) condition for the internal dynamics of the set of interconnected systems (2.15) (Isidori, 1999).

3.1 Polytopic Descriptions of State Space Regions

In this work we will define the set $\mathbb{X}$ to be a polytope in the state space of system (2.27) over the intersections of some hyperplanes:

$$\mathbb{X} = \{x \mid a_k^T x \leq 1, k = 1, ..., n_e\} \ , \tag{3.1}$$

where a_k is a constant n-dimensional vector whose parameters can be calculated, for example, by putting the extremum values of the states in each hyperplane equation $a_{k1}x_{k1} + a_{k2}x_{k2} + ... + a_{kn}x_{kn} = 1$; and n_e is the number of hyperplanes. We will consider that the ellipsoidal Domain of Attraction, which will be defined in Section 3.2, is inside this polytopic set.

One can find the relation between the polytopic set $\mathbb{X} \times \Delta$ and the DAR representation (2.29) of system (2.27). This can be done by showing that the uncertain matrices of DAR system (2.29) are representable by the convex combination of several constant matrices over the vertices of polytopic region defined by $\mathbb{X} \times \Delta$. Since $\mathbb{X} \times \Delta$ is a polytopic region and A_1, A_2, A_3 and Π_1, Π_2, Π_3 are the matrices of affine functions with respect to $(x, \delta p)$, then they belong to polytopes of matrices, i.e. for n_x number of vertices in $\mathbb{X}$ and 2^l number of vertices in Δ:

$$A_1 \in \mathcal{D}_1 = \left\{ A_1(x, \delta p) \in \mathbb{R}^{n \times n} \mid A_1 = \sum_{i=1}^{n_x} \sum_{j=1}^{2^l} \alpha_i \beta_j A_1(x_i, \delta p_j) \right\},$$

$$A_2 \in \mathcal{D}_2 = \left\{ A_2(x, \delta p) \in \mathbb{R}^{n \times n_\pi} \mid A_2 = \sum_{i=1}^{n_x} \sum_{j=1}^{2^l} \alpha_i \beta_j A_2(x_i, \delta p_j) \right\},$$

$$A_3 \in \mathcal{D}_3 = \left\{ A_3(x, \delta p) \in \mathbb{R}^{n \times m} \mid A_3 = \sum_{i=1}^{n_x} \sum_{j=1}^{2^l} \alpha_i \beta_j A_3(x_i, \delta p_j) \right\},$$

$$\Pi_1 \in \mathcal{D}_4 = \left\{ \Pi_1(x, \delta p) \in \mathbb{R}^{n_\pi \times n} \mid \Pi_1 = \sum_{i=1}^{n_x} \sum_{j=1}^{2^l} \alpha_i \beta_j \Pi_1(x_i, \delta p_j) \right\},$$

$$\Pi_2 \in \mathcal{D}_5 = \left\{ \Pi_2(x, \delta p) \in \mathbb{R}^{n_\pi \times n_\pi} \mid \Pi_2 = \sum_{i=1}^{n_x} \sum_{j=1}^{2^l} \alpha_i \beta_j \Pi_2(x_i, \delta p_j) \right\},$$

$$\Pi_3 \in \mathcal{D}_6 = \left\{ \Pi_3(x, \delta p) \in \mathbb{R}^{n_\pi \times m} \mid \Pi_3 = \sum_{i=1}^{n_x} \sum_{j=1}^{2^l} \alpha_i \beta_j \Pi_3(x_i, \delta p_j) \right\},$$

where $\sum_{i=1}^{n_x} \alpha_i = \sum_{j=1}^{2^l} \beta_j = 1$, $\alpha_i, \beta_j \geq 0$. Moreover, $A_1(x_i, \delta p_j), A_2(x_i, \delta p_j), A_3(x_i, \delta p_j)$ and $\Pi_1(x_i, \delta p_j), \Pi_2(x_i, \delta p_j), \Pi_3(x_i, \delta p_j)$ are the valued matrices A_1, A_2, A_3 and Π_1, Π_2, Π_3 in each vertex of $\mathbb{X} \times \Delta$.

To clarify more, consider again Figure 1.1 (page 2) depicted in Section 1.1. The set $\mathbb{X}$ in that figure defines a region whose vertex points x_i, $i \in [1, 4]$ are associated with a polytopic set of states comprising local state trajectories of uncertain nonlinear system (2.29) or (2.27). Analogously, for the DAR system (2.16), which was the approximate I/O feedback linearization of system (2.4), one can assign a state-space and uncertainty polytopic set $\chi \times \Delta$ that includes local trajectories of system (2.16) in (z, ζ)-coordinates. In this respect, the matrices A_1, A_2, A_3 and Π_1, Π_2, Π_3 in (2.16) will be evaluated over the vertices of $\chi \times \Delta$[1]. This notion, for DAR representations, enables to investigate stabilizing LMI conditions, and it will be explained in Chapter 4.

---

[1] Remember that in the context of (z, ζ)-coordinates, the polytope χ is in function of $\Phi(x)$ not x.

3.2   Regional Stability and Domain of Attraction

As previously mentioned in Section 1.1, within the context of regional stability, the DOA is a stabilizing compact region of uncertain nonlinear system (2.27) such that every trajectory initiating inside this region asymptotically converges to the system's origin. In this sense, one can intimately relate the DOA with the Lyapunov theory which is described in (Khalil, 2002). That is if we find a continuously differentiable positive definite function $V(x) : \mathbb{X} \mapsto \mathbb{R}^+$ for the uncertain nonlinear system (2.27) with input saturation and the associated normalized region $\Omega := \{x \in \mathbb{R}^n | V(x) \leq 1\} \subset \mathbb{X}$ inside which the time derivative of $V(x)$ satisfies

$$\dot{V} < 0, \ \ \forall (x, \delta) \in \mathbb{X} \times \Delta, \tag{3.2}$$

one can conclude that any trajectory $x(t)$ of (2.27), initiating inside Ω, approaches the origin (system's equilibrium point) as $t \to \infty$. In this context, Ω is called the DOA of system (2.27).

3.3   Quadratic Stability and Guaranteed Ellipsoidal DOA

Considering the following quadratic Lyapunov function for system (2.27) or its DAR (2.29):

$$V(x) = x^T P x, \ P = P^T > 0, \tag{3.3}$$

the corresponding ellipsoidal DOA is obtained as:

$$\Omega(P, 1) = \{x \in \mathbb{R}^n : x^T P x \leq 1\}. \tag{3.4}$$

If we take the time-derivative of $V(x)$ along the trajectory of the state vector x with the input control command $u(t) = Kx(t)$, $K \in \mathbb{R}^{m \times n}$ being a static linear time invariant feedback of the system states, such that:

$$\dot{V}(x, \delta) = \dot{x}^T P x + x^T P \dot{x}, \tag{3.5}$$

and we show that for all $x \in \Omega(P, 1) \subset \mathbb{X}$ and $\delta \in \Delta$ the time derivative $\dot{V}$ is negative definite, then Ω is called the guaranteed ellipsoidal DOA in the sense of quadratic stability. We will show in Chapter 4 that there exist stabilizing LMI conditions, for the quasi-canonical set of systems (2.15) or its DAR (2.16), in z-coordinates, and for input-saturated system (2.27) or its DAR (2.29), such that $\dot{V} < 0$ is satisfied while the corresponding positive definite matrix P, which

defines the guaranteed ellipsoidal set $\Omega(P, 1)$, together with the static state feedback gain K will be synthesized. Then we will define Semidefinite Programming (SDP) problems subject to those LMIs in order to estimate maximum ellipsoidal DOA inside $\mathbb{X}$ for saturation system and maximum ellipsoidal DOA inside $\mathbb{Z}$, whose definition is the same as (3.1) in z-coordinate, for quasi-canonical system under the condition of input-to-state stability of internal dynamics.

Remark 3.1. It is noteworthy that some previous works have considered more complex shapes for the DOA than the ellipsoidal one aiming to achieve less conservatism (Chesi, 2004a, Chesi et al., 2004, Coutinho et al., 2008, 2009, Coutinho and De Souza, 2013, Trofino and Dezuo, 2014). On the other hand, this leads naturally to more complex Lyapunov candidate functions with respect to the states and uncertain parameters. In this work, the use of simple quadratic Lyapunov candidate functions is important to obtain LMI *synthesis* conditions for the state feedback control gain matrix without relying on any further previous knowledge about the system. This is rather difficult to obtain as one can conclude by studying the referenced papers above where either only analysis conditions are considered (Chesi, 2004a, Chesi et al., 2004, Coutinho et al., 2009, Coutinho and De Souza, 2013, Trofino and Dezuo, 2014), or one has to know in advance an initial estimate of a stabilizing feedback control gain matrix (Coutinho et al., 2008).

Remark 3.2. Another important fact, concerning input-output feedback linearized systems, is that for the quasi-canonical system (2.15) and its DAR (2.16) we want to search for an estimated maximized DOA that is a hyper-ellipsoid in the space of the variables z, and this is not the system state-space because the internal dynamics variables ζ are not account for. To circumvent this problem, we will have to assume that the internal dynamics are input-to-state stable (ISS).

In order to ensure that $\Omega(P, 1)$ is a subset of $\mathbb{X}$ we have to enforce an inequality condition. In this regard, the polytopic set of states $\mathbb{X}$ defined in (3.1) can be, alternatively, rewritten as:

$$2 - a_k^T x - x^T a_k \geq 0, \;\; k = 1, ..., n_e. \tag{3.6}$$

Also according to the quadratic Lyapunov function (3.3) the ellipsoidal DOA $\Omega(P, 1)$ is always a subset of the polytope $\mathbb{X}$ if and only if the following inequalities are satisfied [Lemma 3 in (Rohr et al., 2009)]:

$$1 - a_k^T x - x^T a_k + x^T P x \geq 0, \;\; k = 1, ..., n_e. \tag{3.7}$$

The above inequalities can be rewritten as

$$\begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} 1 & -a_k^T \\ -a_k & P \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0, \quad k = 1, 2, ..., n_e. \tag{3.8}$$

Therefore, the feasibility of the following necessary and sufficient LMIs ensures that $\Omega(P, 1) \subset \mathbb{X}$:

$$\begin{bmatrix} 1 & -a_k^T \\ -a_k & P \end{bmatrix} \geq 0, \quad k = 1, 2, ..., n_e. \tag{3.9}$$

Analogously, we can derive the condition (3.9) for the DOA

$$\Omega_z = \{z \in \mathbb{R}^r : z^T P z \leq 1\}, \tag{3.10}$$

to be the subset of

$$\mathbb{Z} = \{z \mid a_k^T z \leq 1, k = 1, ..., n_e\}, \tag{3.11}$$

in the new state coordinates $z(t)$ of quasi-canonical system.

3.4   Input-to-State Stability of Internal Dynamics

Before investigating the closed-loop regional stability of I/O part of (2.15) or (2.16) one has to guarantee the stability of internal dynamics. Note that since we plan to design an LTI feedback control for the new input $v(t)$, i.e. $v(t) = Kz(t)$, the internal dynamics can be rewritten as $\dot{\zeta} = w(z, \zeta, \delta p)$ which has to remain stable for all $(z, \delta) \in \mathbb{Z} \times \Delta$. This problem can be addressed through the notion of Input-to-State Stability (ISS) of a pair of interconnected systems (2.15) by investigating the stability of internal dynamics subsystem driven by the bounded input $z(t)$ in the presence of uncertainties. Within this context, the definitions of input-to-state stability and ISS-Lyapunov function are recalled here for the internal dynamics of (2.15) as follows (Isidori, 1999):

Definition 3.3. The internal dynamics $\dot{\zeta} = w(z, \zeta, \delta p)$ is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function $\beta(., .)$ and a class $\mathcal{K}$ function $\gamma(.)$ such that, for any input $(z(t), \delta) \in \mathbb{Z} \times \Delta$ and any initial state $\zeta(0) \in \mathbb{R}^{n-r}$, the response $\zeta(t)$ satisfies

$$||\zeta|| \leq \max\{\beta(||\zeta(0)||, t), \gamma(||z||)\}. \tag{3.12}$$

Definition 3.4. A differentiable function $V_\zeta : \mathbb{R}^{n-r} \to \mathbb{R}$ is called an ISS-Lyapunov function for system $\dot\zeta = w(z, \zeta, \delta p)$ if there exist class $\mathcal{K}_\infty$ functions $w_1(.), w_2(.), w_3(.)$, and a class $\mathcal{K}$ function $\alpha(.)$ such that

$$w_1(||\zeta||) \leq V_\zeta(\zeta) \leq w_2(||\zeta||) \;\; \forall \zeta \in \mathbb{R}^{n-r}, \tag{3.13}$$

and

$$||\zeta|| \geq \alpha(||z||) \;\; \Rightarrow \;\; \frac{\partial V_\zeta}{\partial \zeta} w(z, \zeta, \delta p) \leq -w_3(||\zeta||) \;\; \forall \, (\zeta, \delta) \in \mathbb{R}^{n-r} \times \Delta. \tag{3.14}$$

Based on above definitions one can imply that the existence of an ISS-Lyapunov function leads to input-to-state stability of the internal dynamics (Khalil, 2002). In other words, the norm of internal dynamics state vector ζ remains bounded and the bound is characterized by a class $\mathcal{K}$ function of $||z||$. Therefore, if this property is verified, one can investigate regional asymptotic stability of I/O system (2.16) over the vertices of the polytopic set χ characterized by the new state coordinates $\varphi = \Phi(x) = [z^T \;\; \zeta^T]^T$, defined in (2.12), knowing that the ζ trajectories remain bounded according to (3.12).

Remark 3.5. In the case of *asymptotic* stability of I/O system (2.16), according to (3.12), the input-to-state stability of internal dynamics leads to the asymptotic stability as well. This is an interesting result because if one can ensure closed-loop asymptotic stability of the approximate I/O linearized system (2.16), under the condition of input-to-state stability of internal dynamics, it can be concluded that the class $\mathcal{K}$ function $\alpha(||z||)$ tends to zero and therefore (3.14) is true for all $||\zeta|| > 0$, which means the asymptotic convergence of ζ to the origin.

Unfortunately it is not easy to prove, in general, that the internal dynamics of a nonlinear system is ISS. Therefore, this is actually a strong assumption on which we will rely to investigate input-output feedback linearized uncertain systems in the next chapter.

Chapter 4

State Feedback Control Synthesis for Uncertain Nonlinear Dynamical Systems

This chapter investigates the regional asymptotic stability of system (2.15), represented as a DAR system (2.16), which is obtained from the approximate linearization of (2.4). The regional stability analysis is handled by considering polytopes of states and parametric uncertainties following the approaches in (Coutinho et al., 2008, Coutinho and De Souza, 2013, Coutinho et al., 2009, Rohr et al., 2009, Trofino and Dezuo, 2014, Da Silva et al., 2014). To estimate the maximum ellipsoidal DOA for system (2.16), it is applied the Lyapunov direct method as presented in Chapter 3. Then, the DOA estimation is carried out by solving numerically an SDP problem. This approach will also be extended for uncertain nonlinear system described as in (2.27) or its DAR (2.29) with input saturation. In this case a sufficient stabilizing LMI condition is derived based on the poyltopic description of saturation of state feedback input vector as discussed in Section 2.5.1.

4.1 Control Synthesis without Input Saturation

Considering the approximately linearized DAR system in (2.16) for the new input $v(t) = Kz(t)$, the problem is to handle the simultaneous stability analysis of the closed-loop system and control synthesis of a static gain K and also to estimate the corresponding ellipsoidal DOA inside

the state-space polytopic set $\mathbb{Z}$. Similar approach is considered in literature, but instead of a synthesis problem, the control gain K is previously specified before performing any robust stability analysis (Rohr et al., 2009, Trofino and Dezuo, 2014). However, specifying the static gain K beforehand, without taking the system's knowledge into account, imposes conservativeness to the problem. In this section, we will synthesize a stabilizing gain K while solving a convex optimization problem subject to sufficient LMI conditions. If such an optimization problem is feasible the estimation of maximum ellipsoid $\Omega_z \subset \mathbb{Z}$ will be obtained.

4.1.1 Control Design

This section introduces sufficient LMI conditions for approximately I/O linearized system in (2.15) or its DAR (2.16) aiming the synthesis of a linear state feedback control gain. Before explaining the theorem one has to recall S-procedure Lemma from (Boyd et al., 1994).

Lemma 4.1 (S-procedure). *Assume that $T_0, ..., T_p \in \mathbb{R}^{n \times n}$ are symmetric matrices. If there exists $\tau_1 \geq 0, ..., \tau_p \geq 0$ such that $T_0 - \sum\limits_{i=1}^{p} \tau_i T_i < 0$, then $y^T T_0 y < 0$ holds for all $y \in \mathbb{R}^n$ and $y \neq 0$ such that $y^T T_i y \leq 0,\ i = 1, ..., p$.*

Therefore, we are ready to express the following theorem which illustrates the simultaneous stability analysis and control synthesis problem for (2.16) under the existence of ISS Lyapunov function for the internal dynamics in (2.15) as described in Section 3.4.

Theorem 4.2. *Let Δ be a set of admissible uncertainties and assume that there exists an ISS-Lyapunov function for the internal dynamics of (2.15). Consider system (2.16) with the state feedback control $v(t) = Kz(t)$ and the associate closed-loop system:*

$$\dot{z}(t) = (A_1(z, \zeta, \delta p) + A_3(z, \zeta, \delta p)K)z + A_2(z, \zeta, \delta p)\pi, \tag{4.1}$$

$$0 = (\Pi_1(z, \zeta, \delta p) + \Pi_3(z, \zeta, \delta p)K)z + \Pi_2(z, \zeta, \delta p)\pi.$$

If there exist matrices $Q = Q^T > 0$ and $Y \in \mathbb{R}^{m \times r}$, and real scalars $\eta \geq \mu > 0$ such that LMIs

$$\begin{bmatrix} QA_{1i}^T + A_{1i}Q + Y^T A_{3i}^T + A_{3i}Y + \eta A_{2i}A_{2i}^T & Q\Pi_{1i}^T + Y^T\Pi_{3i}^T + \eta A_{2i}(\Pi_{2i} + I)^T \\ \Pi_{1i}Q + \Pi_{3i}Y + \eta(\Pi_{2i} + I)A_{2i}^T & -\mu I + \eta(\Pi_{2i} + I)(\Pi_{2i} + I)^T \end{bmatrix} < 0, \tag{4.2}$$

$$\begin{bmatrix} 1 & -a_k^T Q \\ -Qa_k & Q \end{bmatrix} \geq 0,\ k = 1, 2, ..., n_e, \tag{4.3}$$

are feasible in all vertices of $\chi \times \Delta$ ($n_v = n_\varphi \times 2^l$ vertices) where χ is the polytopic set of new states $\varphi = [z^T \ \zeta^T]^T$ and $A_{j_i}, \Pi_{j_i}, \ j = 1, 2, 3; \ i = 1, ..., n_v$ are the valued matrices A_j and Π_j in each vertex (according to (3.14) the vertices of internal dynamics states are selected such that $\|\zeta\| \geq \alpha(\|z\|)$). Then, in this case, there exists a quadratic Lyapunov function $V(z) = z^T P z, \ P = P^T > 0$ with an associate DOA Ω_z in $\mathbb{Z} \subset \chi$ (Ω_z given in (3.10)) such that for all $z(0)$ starting inside Ω_z and all $\delta \in \Delta$ the trajectories of $z(t)$ asymptotically converge to the origin as $t \to \infty$. In the positive case the control gain is given by $K = YQ^{-1}$.

Proof. For any real scalar $\gamma \leq 1$ one can write $\gamma \leq \frac{\pi^\top \pi}{\pi^\top \pi}$. Then, the following can be inferred from the algebraic equality property in the DAR in (4.1):

$$\gamma \pi^T \pi - \left[(\Pi_{1i} + \Pi_{3i}K)z + (\Pi_{2i} + I)\pi\right]^T \left[(\Pi_{1i} + \Pi_{3i}K)z + (\Pi_{2i} + I)\pi\right] \leq 0. \quad (4.4)$$

The inequality (4.4) can be rewritten as:

$$\begin{bmatrix} z \\ \pi \end{bmatrix}^T \begin{bmatrix} -(\Pi_{1i} + \Pi_{3i}K)^T(\Pi_{1i} + \Pi_{3i}K) & -(\Pi_{1i} + \Pi_{3i}K)^T(\Pi_{2i} + I) \\ -(\Pi_{2i} + I)^T(\Pi_{1i} + \Pi_{3i}K) & \gamma I - (\Pi_{2i} + I)^T(\Pi_{2i} + I) \end{bmatrix} \begin{bmatrix} z \\ \pi \end{bmatrix} \leq 0. \quad (4.5)$$

Taking the time-derivative of the Lyapunov function one gets

$$\dot{V}(z, \zeta, \delta p) = \dot{z}^T P z + z^T P \dot{z} = y^T T_0 y < 0.$$

Denote (4.5) by $y^T T_1 y \leq 0$ with $y = [z^T \ \pi^T]^T$. Notice that considering the S-Procedure (Lemma 4.1) with variable $\tau > 0$, then $T_0 - \tau T_1 < 0$ can be written as:

$$\begin{bmatrix} M_a & PA_{2i} + \tau(\Pi_{1i} + \Pi_{3i}K)^T(\Pi_{2i} + I) \\ A_{2i}^T P + \tau(\Pi_{2i} + I)^T(\Pi_{1i} + \Pi_{3i}K) & -\tau\gamma I + \tau(\Pi_{2i} + I)^T(\Pi_{2i} + I) \end{bmatrix} < 0, \quad (4.6)$$

where $M_a = A_{1i}^T P + PA_{1i} + K^T A_{3i}^T P + PA_{3i}K + \tau(\Pi_{1i} + \Pi_{3i}K)^T(\Pi_{1i} + \Pi_{3i}K)$.

Notice that (4.6) can be rewritten as

$$\begin{bmatrix} A_{1i}^T P + PA_{1i} + K^T A_{3i}^T P + PA_{3i}K & PA_{2i} \\ A_{2i}^T P & -\tau\gamma I \end{bmatrix} - \\ \begin{bmatrix} (\Pi_{1i} + \Pi_{3i}K)^T \\ (\Pi_{2i} + I)^T \end{bmatrix} (-\frac{1}{\tau})^{-1} \begin{bmatrix} (\Pi_{1i} + \Pi_{3i}K) & (\Pi_{2i} + I) \end{bmatrix} < 0.$$

Pre- and post multiplying the above inequality by $\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}$ with $Q = P^{-1}$ and $Y = KQ$ it follows that

$$
\begin{bmatrix} QA_{1i}^T + A_{1i}Q + Y^T A_{3i}^T + A_{3i}Y & A_{2i} \\ A_{2i}^T & -\tau\gamma I \end{bmatrix} -
$$
$$
\begin{bmatrix} Q\Pi_{1i}^T + Y^T \Pi_{3i}^T \\ (\Pi_{2i} + I)^T \end{bmatrix} (-\tfrac{1}{\tau})^{-1} \begin{bmatrix} \Pi_{1i}Q + \Pi_{3i}Y & (\Pi_{2i} + I) \end{bmatrix} < 0,
$$

which after applying Schur complement leads to:

$$
\begin{bmatrix} QA_{1i}^T + A_{1i}Q + A_{3i}Y + Y^T A_{3i}^T & A_{2i} & Q\Pi_{1i}^T + Y^T \Pi_{3i}^T \\ A_{2i}^T & -\tau\gamma I & (\Pi_{2i} + I)^T \\ \Pi_{1i}Q + \Pi_{3i}Y & (\Pi_{2i} + I) & -\tfrac{1}{\tau}I \end{bmatrix} < 0 . \tag{4.7}
$$

A necessary condition for the feasibility of this inequality is that $\gamma > 0$, therefore $0 < \gamma \le 1$. Rearranging some elements of (4.7) by post multiplying it by the nonsingular matrix $M = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$ and pre- multiplying it by $M^{-1} = M$ (notice that this rearrangement does not affect the feasibility of (4.7)(VanAntwerp and Braatz, 2000)) it follows that

$$
\begin{bmatrix} QA_{1i}^T + A_{1i}Q + A_{3i}Y + Y^T A_{3i}^T & Q\Pi_{1i}^T + Y^T \Pi_{3i}^T & A_{2i} \\ \Pi_{1i}Q + \Pi_{3i}Y & -\tfrac{1}{\tau}I & (\Pi_{2i} + I) \\ A_{2i}^T & (\Pi_{2i} + I)^T & -\tau\gamma I \end{bmatrix} < 0. \tag{4.8}
$$

Applying again the Schur complement on (4.8), the following is obtained

$$
\begin{bmatrix} QA_{1i}^T + A_{1i}Q + Y^T A_{3i}^T + A_{3i}Y & Q\Pi_{1i}^T + Y^T \Pi_{3i}^T \\ \Pi_{1i}Q + \Pi_{3i}Y & -\tfrac{1}{\tau}I \end{bmatrix} -
$$
$$
\begin{bmatrix} A_{2i} \\ (\Pi_{2i} + I) \end{bmatrix} (-\tau\gamma I)^{-1} \begin{bmatrix} A_{2i}^T & (\Pi_{2i} + I)^T \end{bmatrix} < 0. \tag{4.9}
$$

Therefore, defining $\eta = \tfrac{1}{\tau\gamma}$ and $\mu = \tfrac{1}{\tau}$ in the above inequality LMI (4.2) is obtained. Notice also that LMI (4.3) explains the DOA $\Omega_z \subset \mathbb{Z}$ in z-coordinates and can be derived by pre- and post multiplying (3.9) by $\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$. $\qquad\square$
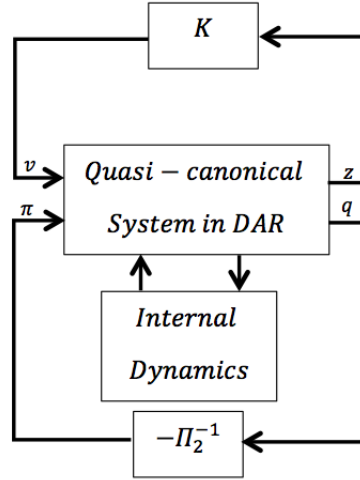
FIGURE 4.1: Differential Algebraic Representation of feedback linearized closed-loop system.

As a matter of fact, the feasibility of the set of LMIs (4.2) and (4.3) over the vertices of $\chi \times \Delta$ leads to finding a DOA Ω_z, which is characterized by $P = Q^{-1}$, in z-coordinates. In this context, one can define the following SDP problem in order to estimate the maximum ellipsoidal DOA within the polytopic region in state space:

$$\max \left\{ \log \left(\det \left(Q \right) \right) \right\} \quad \text{s.t.} \quad (4.2) \text{ and } (4.3). \tag{4.10}$$

The following corollary states the regional asymptotic stability of the system in (2.15):

Corollary 4.3. *Assume that there exists a feasible solution for the SDP problem* (4.10) *and there exists an ISS-Lyapunov function for* $\dot{\zeta} = w(z, \zeta, \delta p)$. *Then* (2.15) *is locally asymptotically stable for all* $\delta \in \Delta$.

Proof. The proof follows in a straightforward way from Remark 3.5 and Theorem 4.2. □

The block diagram of the closed-loop quasi-linearized system (4.1) together with the associate internal dynamics is depicted in Figure 4.1 in which the control vector v together with the nonlinear uncertain vector π are the inputs and the state vector z together with the vector $q = (\Pi_1 + \Pi_3 K)z$ are the outputs.

4.2 Control Synthesis with Input Saturation

In this section, the simultaneous stability analysis and control synthesis of uncertain nonlinear system (2.27) or its DAR (2.29), with saturated input, is investigated. The problem is to find an appropriate static control gain K, for $\text{sat}(u(t)) = \text{sat}(Kx(t))$ and to estimate the corresponding DOA $\Omega(P,1) = \{x \in \mathbb{R}^n | x^T P x \leq 1\}$ in the context of quadratic Lyapunov candidate function. In the sequel we recall the definition of linear annihilators, which leads to introducing auxiliary decision variable and obtaining more relaxed LMI conditions.

4.2.1 Linear Annihilators

Linear annihilators are useful tools that have been recently proposed in (Trofino and Dezuo, 2014), and employed in (Coutinho et al., 2008, Coutinho and De Souza, 2013). It has been shown that by using linear annihilators one can obtain less conservative stabilizing LMIs, although linear annihilators are not unique. However, the larger dimension of an annihilator the less conservative the corresponding result. The definition of this large enough annihilator is recalled here for the states of system (2.27) as introduced in (Trofino and Dezuo, 2014).

Definition 4.4. (Trofino and Dezuo, 2014)[Linear annihilator] For system (2.27) the matrix $\mathcal{X}_x : \mathbb{R}^n \mapsto \mathbb{R}^{q \times n}$ is a linear annihilator of state vector $x(t)$ if $\mathcal{X}_x x(t) = 0$ and $\mathcal{X}_x$ is linear with respect to $x(t)$. A possible $\mathcal{X}_x$ can be obtained as:

$$\mathcal{X}_x(x) = \begin{bmatrix} \Phi_1(x) & \Psi_1(x) \\ \vdots & \vdots \\ \Phi_{n-1}(x) & \Psi_{n-1}(x) \end{bmatrix}, \tag{4.11}$$

where

$$\Psi_i(x) = -x_i I_{n-i}, \ i \in [1, n-1],$$

$$\Phi_1(x) = [x_2 \ldots x_n]^T, \ \Phi_i(x) = \begin{bmatrix} & & x_{i+1} \\ 0_{(n-i) \times (i-1)} & & \vdots \\ & & x_n \end{bmatrix}, \ i \in [2, n-1]$$

and, therefore, the number of rows of this possible linear annihilator is $q = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$.

Following the approach in (Trofino and Dezuo, 2014) the combination of linear annihilators and Finsler Lemma can be used to introduce a new auxiliary decision variable and get more relaxed LMI conditions.

4.2.2 Control Synthesis for the DAR of Input-saturated Nonlinear System

The idea is to consider that the closed-loop system (2.29) has an input saturation which can be replaced with the convex combination of the input $u(t) = Kx(t)$ and a linear feedback of sates $\tilde{u}(t) = Hx(t)$, where $K, H \in \mathbb{R}^{m \times n}$ are two unknown matrices to be synthesized. That is, in view of Lemma 2.3 and putting $u(t) = Kx(t)$ and $\tilde{u}(t) = Hx(t)$, as long as it is possible to find a matrix H such that $|h_j x| \leq u_{0_j}, j \in [1, m]$, with h_j the jth row of the auxiliary matrix H, the saturation input $\text{sat}(Kx(t))$ can be replaced by the convex hull of $D_s Kx(t)$ and $D_s^- Hx(t)$ as:

$$\text{sat}(Kx(t)) = \sum_{s=1}^{2^m} \alpha_s (D_s K + D_s^- H)x, \tag{4.12}$$

where $\alpha = [\alpha_1, ..., \alpha_N]^T$, $N = 2^m$, is a real parameter vector belonging to the simplex $\Lambda_N := \left\{ \lambda \in \mathbb{R}^N : \sum_{s=1}^{N} \lambda_s = 1, \lambda_s \geq 0 \right\}$. Accordingly, the polytopic description of the input-saturated closed-loop DAR (2.29) becomes:

$$\dot{x}(t) = \left(A_1 + A_3 \sum_{s=1}^{2^m} \alpha_s (D_s K + D_s^- H) \right) x + A_2 \pi,$$

$$0 = \left(\Pi_1 + \Pi_3 \sum_{s=1}^{2^m} \alpha_s (D_s K + D_s^- H) \right) x + \Pi_2 \pi, \tag{4.13}$$

where the matrices A_1, A_2, A_3, Π_1, Π_2 and Π_3 can be replaced by the convex combinations of matrices over the vertices of the polytopic region $\mathbb{X} \times \Delta$ as it was explained in Section 3.1. Using the Lyapunov theory it can be proven that in the context of DOA $\Omega(P, 1) \subset \mathbb{X}$ the stability of 2^m nonlinear uncertain systems results in the stability of system (2.29) for $u(t) = Kx(t)$.

Theorem 4.5. *For a given ellipsoid $\Omega(P, 1) \subset \mathbb{X}$ assume that there exist $K, H \in \mathbb{R}^{m \times n}$ such that:*

$$\Omega(P, 1) \subset \left\{ x \in \mathbb{R}^n : |h_j x| \leq u_{0_j}, j \in [1, m] \right\}, \tag{4.14}$$

and

$$2x^T P \Big[\big(A_1 + A_3 (D_s K + D_s^- H) \big) x + A_2 \pi \Big] < 0, \quad \forall s \in [1, 2^m], \ (x, \delta) \in \Omega(P, 1) \times \Delta \backslash \{(0, \delta)\}.$$
$$(4.15)$$

Then, $\Omega(P, 1)$ is a DOA for system (2.29) *with $u(t) = Kx(t)$, and a corresponding quadratic Lyapunov function is given by*

$$V(t) = x^T P x. \tag{4.16}$$

Proof. Consider the quadratic Lyapunov candidate function $V = x^T P x$ for system (2.29) with $u(t) = Kx(t)$, then

$$\dot{V} = 2x^T P \big[A_1 x + A_2 \pi + A_3 \text{sat}(Kx(t)) \big].$$

By assumption for all $x \in \Omega(P, 1)$ we know that $|h_j x| \leq u_{0_j}$. According to Lemma 2.3 there exists an $\alpha \in \Lambda_N$ such that (4.12) holds and then we can replace $\text{sat}(Kx(t))$ in $\dot{V}$ as:

$$\dot{V} = 2x^T P \Big[A_1 x + A_2 \pi + A_3 \sum_{s=1}^{2^m} \alpha_s (D_s K + D_s^- H) x \Big]$$

$$= \sum_{s=1}^{2^m} \alpha_s \Big\{ 2x^T P \Big[\big(A_1 + A_3 (D_s K + D_s^- H) \big) x + A_2 \pi \Big] \Big\}.$$

Since $\alpha_s \geq 0, \ \forall s \in [1, 2^m]$, we can conclude from assumption (4.15) that $\dot{V} < 0$ for all $(x, \delta) \in \Omega(P, 1) \times \Delta \backslash \{(0, \delta)\}$ and $\Omega(P, 1)$ is a DOA for system (2.29). $\qquad \square$

Theorem 4.5 implies that if $\Omega(P, 1)$ is a DOA for 2^m systems:

$$\dot{x}(t) = \Big(A_1 + A_3 (D_s K + D_s^- H) \Big) x + A_2 \pi,$$
$$0 = \Big(\Pi_1 + \Pi_3 (D_s K + D_s^- H) \Big) x + \Pi_2 \pi, \tag{4.17}$$

with $s \in [1, 2^m]$, then it will also be a DOA for system (2.29) (or equivalently (2.27)) with $u(t) = Kx(t)$. Therefore, Theorem 4.5 is an instrumental tool to obtain the largest possible ellipsoidal DOA for the original system.

Before explaining the main result, one has to recall part of the well-known Finsler Lemma (de Oliveira and Skelton, 2002).

Lemma 4.6 (Finsler). *Let $\xi \in \mathbb{R}^{n_\xi}$, $\Xi \in \mathbb{R}^{n_\xi \times n_\xi}$ and $\mathcal{H} \in \mathbb{R}^{r \times n_\xi}$ (rank($\mathcal{H}$) $< n_\xi$). The following two statements are equivalent:*

1. $\xi^T \Xi \xi < 0, \quad \forall \, \mathcal{H}\xi = 0, \ \xi \neq 0.$

2. $\exists \mathcal{N} \in \mathbb{R}^{n_\xi \times r} : \Xi + \mathcal{N}\mathcal{H} + \mathcal{H}^T \mathcal{N}^T < 0.$

Now, we can use Theorem 4.5 and Lemma 4.6 to present the main theorem.

Theorem 4.7. *Consider the set of closed-loop systems given in* (4.17). *For each system let Δ be a set of admissible uncertainties and suppose that for a given positive real number γ there exist matrices $Q_1 = Q_1^T > 0, Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{n_\pi \times n_\pi}$, $Y_1, Y_2 \in \mathbb{R}^{m \times n}$, and a real scalar $\eta > 0$ such that the LMIs*

$$
\begin{bmatrix}
\mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} & Q_1 \mathcal{X}_x^T \\[2ex]
* & -2\gamma Q_1 & \gamma A_{2i} Q_2^T & 0 \\[2ex]
* & * & \Pi_{2i} Q_2^T + Q_2 \Pi_{2i}^T & 0 \\[2ex]
* & * & * & -\frac{\eta}{2} I
\end{bmatrix} < 0 , \quad s \in [1, 2^m],\ i \in [1, n_v],\ n_v = n_x \times 2^l
$$

$$(4.18)$$

$$
\mathcal{A}_{11} = A_{1i} Q_1 + A_{3i}(D_s Y_1 + D_s^- Y_2) + Q_1 A_{1i}^T + (Y_1^T D_s + Y_2^T D_s^-)A_{3i}^T,
$$

$$
\mathcal{A}_{12} = \gamma Q_1 A_{1i}^T + \gamma(Y_1^T D_s + Y_2^T D_s^-)A_{3i}^T,
$$

$$
\mathcal{A}_{13} = A_{2i} Q_2^T + Q_1 \Pi_{1i}^T + (Y_1^T D_s + Y_2^T D_s^-)\Pi_{3i}^T,
$$

$$
\begin{bmatrix} u_{0_j}^2 & Y_{2_j} \\ Y_{2_j}^T & Q_1 \end{bmatrix} \geq 0, \ j \in [1, m], \ \text{with } Y_{2_j} \text{ the } j^{th} \text{ row of } Y_2, \tag{4.19}
$$

$$
\begin{bmatrix} 1 & a_k^T Q_1 \\ Q_1 a_k & Q_1 \end{bmatrix} \geq 0, \ k \in [1, n_e], \tag{4.20}
$$

are feasible in all vertices of $\Delta \times \mathbb{X}$, where A_{1i}, A_{2i}, A_{3i} and $\Pi_{1i}, \Pi_{2i}, \Pi_{3i}$ are the valued matrices A_1, A_2, A_3 and Π_1, Π_2, Π_3 in each vertex. In this case, there exists a quadratic Lyapunov function (4.16) *with $P = Q_1^{-1}$, and the control gain given by $K = Y_1 Q_1^{-1}$ for the original system* (2.27) *with $u(t) = Kx(t)$, such that, for all $x(0)$ starting inside $\Omega(P, 1)$ and all $\delta \in \Delta$, the trajectory of $x(t)$ approaches the origin as $t \to \infty$ and $\Omega(P, 1)$ is also a DOA for system* (2.27).

Proof. Consider the Lyapunov candidate function (4.16) for model in (4.17). Thus, the idea is to rewrite the negative definite condition for the time-derivative of the Lyapunov function as follows:

$$
\dot{V} = \dot{x}^T P x + x^T P \dot{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \pi \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \pi \end{bmatrix} < 0. \qquad (4.21)
$$

On the other hand, using the equality property of DAR form in (4.17) and the linear annihilator obtained from the formula (4.11) we have:

$$
\begin{bmatrix} \mathcal{X}_x & 0 & 0 \\ A_{1i} + A_{3i}(D_s K + D_s^- H) & -I & A_{2i} \\ \Pi_{1i} + \Pi_{3i}(D_s K + D_s^- H) & 0 & \Pi_{2i} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \pi \end{bmatrix} = 0, \qquad (4.22)
$$

Now, considering $\xi = [x \ \dot{x} \ \pi]^T$ one can compare (4.21) and (4.22) with the first statement of Lemma 4.6. Therefore there exists a block matrix-variable

$$
\mathcal{N} = \begin{bmatrix} N_{11}^{n \times q} & N_{12}^{n \times n} & N_{13}^{n \times n_\pi} \\ \\ N_{21}^{n \times q} & N_{22}^{n \times n} & N_{23}^{n \times n_\pi} \\ \\ N_{31}^{n_\pi \times q} & N_{32}^{n_\pi \times n} & N_{33}^{n_\pi \times n_\pi} \end{bmatrix} \in \mathbb{R}^{(2n+n_\pi) \times (q+n+n_\pi)},
$$

such that:

$$
\begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{N} \begin{bmatrix} \mathcal{X}_x & 0 & 0 \\ A_{1i} + A_{3i}(D_s K + D_s^- H) & -I & A_{2i} \\ \Pi_{1i} + \Pi_{3i}(D_s K + D_s^- H) & 0 & \Pi_{2i} \end{bmatrix}
$$
$$
+ \begin{bmatrix} \mathcal{X}_x & 0 & 0 \\ A_{1i} + A_{3i}(D_s K + D_s^- H) & -I & A_{2i} \\ \Pi_{1i} + \Pi_{3i}(D_s K + D_s^- H) & 0 & \Pi_{2i} \end{bmatrix}^T \mathcal{N}^T < 0,
$$

which is expanded as:

$$
\begin{bmatrix}
\mathcal{B}_{11}^1 & \mathcal{B}_{12}^1 & \mathcal{B}_{13}^1 \\[2em]
* & -N_{22} - N_{22}^T & N_{22}A_{2i} + N_{23}\Pi_{2i} - N_{32}^T \\[2em]
* & * & N_{32}A_{2i} + N_{33}\Pi_{2i} + A_{2i}^T N_{32}^T + \Pi_{2i}^T N_{33}^T
\end{bmatrix} < 0,
$$

$$
\mathcal{B}_{11}^1 = N_{11}\mathcal{X}_x + \mathcal{X}_x^T N_{11}^T + N_{12}\big(A_{1i} + A_{3i}(D_sK + D_s^- H)\big) + \big(A_{1i} + A_{3i}(D_sK +
$$
$$
D_s^- H)\big)^T N_{12}^T + N_{13}\big(\Pi_{1i} + \Pi_{3i}(D_sK + D_s^- H)\big) + \big(\Pi_{1i} + \Pi_{3i}(D_sK + D_s^- H)\big)^T N_{13}^T,
$$
$$
\mathcal{B}_{12}^1 = P - N_{12} + \mathcal{X}_x^T N_{21}^T + \big(A_{1i} + A_{3i}(D_sK + D_s^- H)\big)^T N_{22}^T + \big(\Pi_{1i} + \Pi_{3i}(D_sK + D_s^- H)\big)^T N_{23}^T,
$$
$$
\mathcal{B}_{13}^1 = \mathcal{X}_x^T N_{31}^T + N_{12}A_{2i} + N_{13}\Pi_{2i} + \big(A_{1i} + A_{3i}(D_sK + D_s^- H)\big)^T N_{32}^T + \big(\Pi_{1i} +
$$
$$
\Pi_{3i}(D_sK + D_s^- H)\big)^T N_{33}^T.
$$

Clearly the above inequality is not an LMI. However, by choosing appropriately the block matrix $\mathcal{N}$ one can obtain an LMI. Therefore, we can choose some elements of $\mathcal{N}$ as

$$
N_{21}, N_{13}, N_{23}, N_{31}, N_{32} = 0, \ N_{12} = P, \ N_{22} = \gamma P,
$$

where γ is a real positive number to be determined. Then, the following inequality is obtained:

$$
\begin{bmatrix}
\mathcal{B}_{11}^2 & \mathcal{B}_{12}^2 & \mathcal{B}_{13}^2 \\[2em]
* & -2\gamma P & \gamma P A_{2i} \\[2em]
* & * & N_{33}\Pi_{2i} + \Pi_{2i}^T N_{33}^T
\end{bmatrix} < 0,
$$

$$
\mathcal{B}_{11}^2 = N_{11}\mathcal{X}_x + \mathcal{X}_x^T N_{11}^T + P\big(A_{1i} + A_{3i}(D_sK + D_s^- H)\big) + \big(A_{1i} + A_{3i}(D_sK + D_s^- H)\big)^T P,
$$
$$
\mathcal{B}_{12}^2 = \gamma\big(A_{1i} + A_{3i}(D_sK + D_s^- H)\big)^T P,
$$
$$
\mathcal{B}_{13}^2 = PA_{2i} + \big(\Pi_{1i} + \Pi_{3i}(D_sK + D_s^- H)\big)^T N_{33}^T.
$$

It should be noted that since the matrix Π_{2i} is full-rank and invertible, from the feasibility of the term $N_{33}\Pi_{2i} + \Pi_{2i}^T N_{33}^T < 0$ in above inequality, one can infer that N_{33} is non-singular and invertible as well. Therefore, by Pre- and post multiplying the above inequality by $\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & P^{-1} & 0 \\ 0 & 0 & N_{33}^{-1} \end{bmatrix}$

and $\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & P^{-1} & 0 \\ 0 & 0 & N_{33}^{-1} \end{bmatrix}^T$ respectively, we have:

$$\begin{bmatrix} \mathcal{B}_{11}^3 & \mathcal{B}_{12}^3 & \mathcal{B}_{13}^3 \\ * & -2\gamma P^{-1} & \gamma A_{2i} N_{33}^{-1^T} \\ * & * & \Pi_{2i}N_{33}^{-1^T} + N_{33}^{-1}\Pi_{2i}^T \end{bmatrix} < 0,$$

$$\mathcal{B}_{11}^3 = P^{-1}(N_{11}\mathcal{X}_x + \mathcal{X}_x^T N_{11}^T)P^{-1} + A_{1i}P^{-1} + A_{3i}(D_s K + D_s^- H)P^{-1} + P^{-1}A_{1i}^T + $$
$$P^{-1}(D_s K + D_s^- H)^T A_{3i}^T,$$
$$\mathcal{B}_{12}^3 = \gamma P^{-1}A_{1i}^T + \gamma P^{-1}(D_s K + D_s^- H)^T A_{3i}^T,$$
$$\mathcal{B}_{13}^3 = A_{2i}N_{33}^{-1^T} + P^{-1}\Pi_{1i}^T + P^{-1}(D_s K + D_s^- H)^T\Pi_{3i}^T.$$

Now, defining $N_{11} = \frac{1}{\eta}\mathcal{X}_x^T$ $(\eta > 0)$ and using the change of variables $P^{-1} = Q_1, N_{33}^{-1} = Q_2, KQ_1 = Y_1, HQ_1 = Y_2$ we obtain:

$$\begin{bmatrix} \mathcal{B}_{11}^4 & \mathcal{B}_{12}^4 & \mathcal{B}_{13}^4 \\ * & -2\gamma Q_1 & \gamma A_{2i}Q_2^T \\ * & * & \Pi_{2i}Q_2^T + Q_2\Pi_{2i}^T \end{bmatrix} < 0,$$

$$\mathcal{B}_{11}^4 = \frac{2}{\eta}Q_1\mathcal{X}_x^T\mathcal{X}_x Q_1 + A_{1i}Q_1 + A_{3i}(D_s Y_1 + D_s^- Y_2) + Q_1 A_{1i}^T + (Y_1^T D_s + Y_2^T D_s^-)A_{3i}^T,$$
$$\mathcal{B}_{12}^4 = \gamma Q_1 A_{1i}^T + \gamma(Y_1^T D_s + Y_2^T D_s^-)A_{3i}^T,$$
$$\mathcal{B}_{13}^4 = A_{2i}Q_2^T + Q_1\Pi_{1i}^T + (Y_1^T D_s + Y_2^T D_s^-)\Pi_{3i}^T.$$

Finaly, applying the Schur complement to the inequality above, one gets LMI (4.18).

In order to guarantee that the DOA satisfies assumption (4.14) in Theorem 4.5, one should solve the following optimization problem:

$$min\{V\} \ \ s.t. \ \ h_j x = \pm u_{0_j}, \ j \in [1, m], \tag{4.23}$$

with $V = x^T P x$. If the minimum value $V_{\min} = \min\{V\}$ satisfies $V_{\min} \geq 1$ in this case the ellipsoidal region $\Omega(P, 1)$ will be entirely inside the space defined by the hyperplanes $h_j x = \pm u_{0_j}$. Applying the Lagrange method one can write the corresponding Lagrangian function as:

$$L := x^T P x + \lambda(h_j x \pm u_{0_j}) \tag{4.24}$$

where λ is the Lagrange multiplier. Therefore, the minimum of (4.24) can be found by considering:

$$\nabla L = 0 \Rightarrow x_* = -\tfrac{\lambda}{2} P^{-1} h_j^T,$$
$$h_j x_* \pm u_{0_j} = 0 \Rightarrow \lambda_* = \pm 2 u_{0_j} (h_j P^{-1} h_j^T)^{-1}.$$

Replacing λ_* and x_* in (4.24) we have:

$$L_{\min} = u_{0_j}^2 (h_j P^{-1} h_j^T)^{-1} \geq 1,$$

which, by applying Schur complement, is equivalent to:

$$\begin{bmatrix} u_{0_j}^2 & h_j \\ h_j^T & P \end{bmatrix} \geq 0.$$

Pre- and post multiplying the above inequality by $\begin{bmatrix} 1 & 0 \\ 0 & P^{-1} \end{bmatrix}$ the set of LMIs (4.19) are obtained where $Q_1 = P^{-1}$ and $Y_{2_j} = h_j Q_1$. With the same reasoning to meet the assumption $\Omega(P, 1) \subset \mathbb{X}$, where $\mathbb{X}$ is defined in (3.1), the problem

$$min\{\tilde{V}\} \ \ s.t. \ \ a_k^T x = 1, \ k \in [1, n_e] \tag{4.25}$$

with $\tilde{V} = x^T P x$ should be solved and it will result in the following inequality:

$$(a_k^T P^{-1} a_k)^{-1} \geq 1 \equiv \begin{bmatrix} 1 & a_k^T \\ a_k & P \end{bmatrix} \geq 0,$$

where by pre- and post multiplying it by $\begin{bmatrix} 1 & 0 \\ 0 & P^{-1} \end{bmatrix}$ the set of LMIs (4.20) are obtained. $\qquad\square$

Remark 4.8. Note that in light of Theorem 4.5 the existence of a Lyapunov function as given in (4.16) and the feasibility of the inequalities (4.18)-(4.20) for the set of systems in (4.17) leads to the same reasoning for systems (2.29) and (2.27). In other words, one can conclude from Theorem 4.7 that the feasibility of (4.18) leads to the existence of matrices P, K and H such that (4.15) is true and also the feasibility of (4.19) and (4.20) lead to the fact that the corresponding DOA $\Omega(P, 1)$ is always inside both the state polytopic set $\mathbb{X}$ and the set of state vectors satisfying $|h_j x| \leq u_{0_j}$. Therefore, in light of Theorem 4.5 the set $\Omega(P, 1)$ is a guaranteed ellipsoidal domain of attraction for the system (2.27) within which all states trajectories asymptotically converge to the origin.

Now, an SDP problem subject to LMIs (4.18) to (4.20) can be solved by seeking the largest $\Omega(P, 1)$. In this regard, to maximize the volume of $\Omega(P, 1)$ one can define an objective function such as $\log\left(\det\left(Q_1\right)\right)$, resulting in the following convex optimization problem:

$$\max\left\{\log\left(\det\left(Q_1\right)\right)\right\} \quad \text{s.t.} \quad (4.18) - (4.20). \tag{4.26}$$

Remark 4.9. Note that the number of LMIs (4.18) to be solved by numerical solvers is equal to $n_x \times 2^{l+m}$ because their feasibility should be evaluated over the set of systems (4.17). Therefore, for MIMO dynamical systems with many states and control inputs the problem can become computationally prohibitive.

Chapter 5

Numerical Examples

5.1   Input/Output Linearizable System with Internal Dynamics

In this first example, the idea is to solve a control problem for the set of uncertain systems as in (2.15) consisting of approximately I/O linearized subsystems in z-coordinates and considering the existence of an ISS Lyapunov function for the internal dynamics. For that, consider the I/O linearizable MIMO system given in (2.23) which following (2.15) has the approximately I/O linearized form in (2.25) with corresponding leftover dynamos (internal dynamics) in (2.26). In this case the DAR of the approximately I/O linearized systems (2.25) has the form as in (2.16) with:

$$\pi = \begin{bmatrix} \delta_1 \zeta z_3 & \delta_1 z_3 & \delta_1 v_1 \end{bmatrix}^\top,$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2\zeta - 2z_3 & -6z_3 - 4z_1 - 1 & 2z_3 \end{bmatrix},$$

$$\Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_1 \\ 0 & 0 & 0 \end{bmatrix}, \Pi_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \delta_1 & 0 \end{bmatrix}, \Pi_2 = \begin{bmatrix} -1 & \zeta & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Before solving the SDP problem (4.10) one has to verify the input-to-state stability of (2.26). Accordingly, consider an ISS-Lyapunov candidate function $V_\zeta(\zeta) = \frac{1}{2}\zeta^2$ which satisfies (3.13).

Then, we have

$$\dot{V}_\zeta = \frac{\partial V_\zeta(\zeta)}{\partial \zeta} w(z, \zeta, \delta p) = \zeta(z_2 - (1 + \delta_2)\zeta) \leq ||\zeta||\,||z_2|| - (1 + \delta_2)||\zeta||^2.$$

Now, one should seek if there exists a class $\mathcal{K}_\infty$ function $w_3(||\zeta||)$ such that (3.14) is satisfied. To do so, pick any real scalar $0 < \epsilon < 1$ and, according to (3.14), set

$$\alpha(||z||) = \frac{1}{1 - \epsilon}||z_2||.$$

Now, if $||\zeta|| \geq \alpha(||z||)$ then $||\zeta||\,||z_2|| \leq (1 - \epsilon)||\zeta||^2$ and

$$\frac{\partial V_\zeta(\zeta)}{\partial \zeta} w(z, \zeta, \delta p) \leq -(\epsilon + \delta_2)||\zeta||^2,$$

in which for all $|\delta_2| < \epsilon$ the internal dynamics (2.26) is input-to-state stable. Therefore, in order to solve the SDP problem (4.10) for the closed-loop I/O linearized system (4.1) one should determine the vertices of the polytopic set χ such that $0 < \epsilon \leq 1 - \frac{||z_2||}{||\zeta||}$. In this regard, and checking the feasibility of LMIs (4.2) and (4.3), the following bounds for the polytopic sets of states and uncertainties are obtained as:

$$|z_1| \leq 0.2, \quad |z_2| \leq 0.2, \quad |z_3| \leq 0.07, \quad |\zeta| \leq 0.67,$$
$$|\delta_1| \leq 0.04, \quad |\delta_2| \leq 0.7015,$$

such that by solving SDP (4.10) the estimated maximum ellipsoidal DOA and the synthesized linear feedback control gain matrix K are calculated as:

$$\Omega_z = \left\{ z \in \mathbb{R}^3 \,|\, z^T \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0.0036 \\ 0 & 0.0036 & 204.0816 \end{bmatrix} z \leq 1 \right\},$$

$$K = \begin{bmatrix} -0.0099 & 0 & -0.0012 \\ -0.0001 & -0.1227 & -27.7746 \end{bmatrix},$$

which guarantees the regional asymptotic stability of the approximately I/O linearized system in (2.25). Therefore, according to Corollary 4.3, for all $|\zeta| \leq 0.67$, the internal dynamics (2.26) is asymptotically stable as well. Figure 5.1 shows the output trajectories of the system (2.23) with different initial conditions within the border of the projected DOA in $z_1 - z_2$ plane together
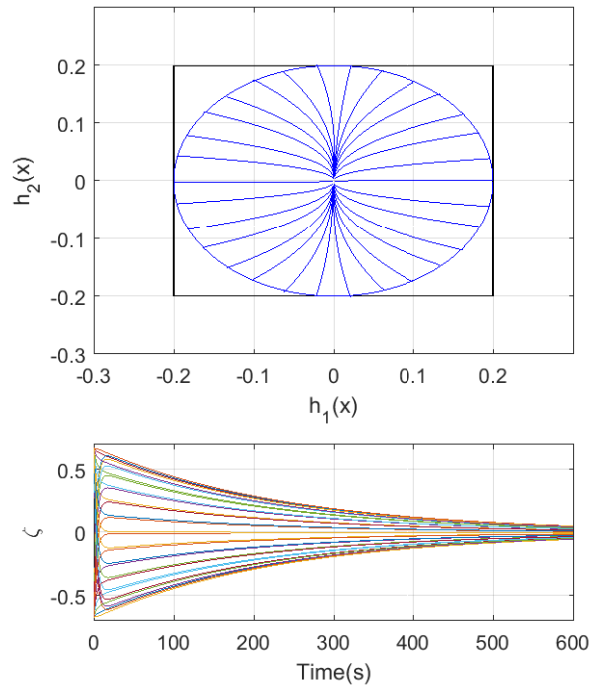
FIGURE 5.1: Estimated projected DOA in the $z_1 - z_2$ plane (upper plot) and output trajectories together with the time responses of internal dynamics (lower plot).
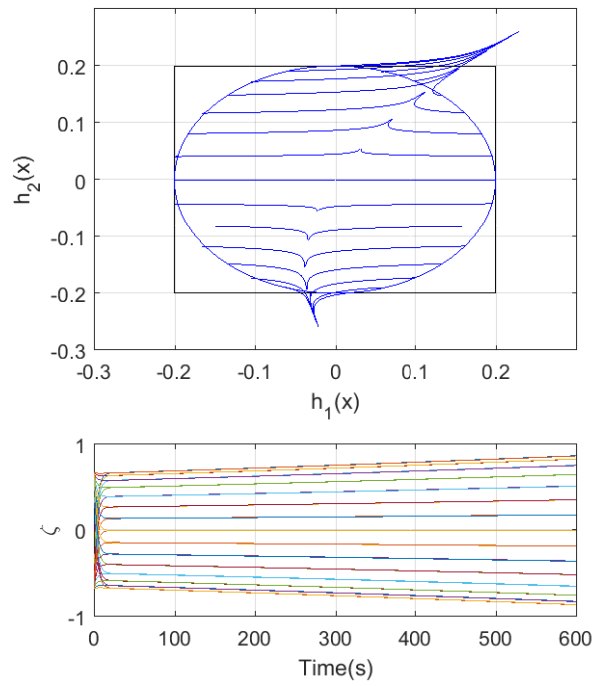

FIGURE 5.2: Output trajectories and time responses of internal dynamics for $\delta_1 = 1.1$.

with the time responses of internal dynamics. The results illustrate the asymptotic convergence of system states within the polytopic set χ for the admissible parametric uncertainties inside Δ.

In order to analyze the robustness performance of the closed-loop system, several simulations are performed by gradually increasing the bound of uncertainty δ_1 such that unstable conditions occur. Therefore, as depicted in Figure 5.2, the system responses go unstable if $|\delta_1| \geq 1.1$.

5.2 Feedback Linearized Inverted Pendulum without Input Saturation

This section checks the stability of the approximately linearized inverted pendulum given in (2.22) in z-coordinates, (see section 4.1), and gives an estimate to its DOA in the new coordinates for the designed control law. A comparison with reference (Rohr et al., 2009) is also presented.

The data for the DAR of the inverted pendulum in (2.22) are:

$$\pi = \begin{bmatrix} \frac{z_1}{1+\delta_1} & \frac{z_2}{1+\delta_1} & \frac{z_1 z_2 \delta_1}{(1+\delta_1)(1+z_1^2)} & \frac{z_1 \delta_1}{1+z_1^2} & \frac{z_1^2 \delta_1}{1+z_1^2} & \frac{v}{1+\delta_1} \end{bmatrix}^T, A_1 = A_c,$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{g}{l}\delta_1 & -\frac{b_0}{M_0}\delta_2 & 2z_2 & 0 & 0 & -\delta_1 \end{bmatrix}, A_3 = b_c, \Pi_1 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.006 \\ 0 & 0 \\ 0.02\delta_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \Pi_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.002 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} -0.02(1+\delta_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.006(1+\delta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.02(1+\delta_1) & 0.02z_2 & 0 & 0 \\ 0 & 0 & 0 & -0.02 & -0.02z_1 & 0 \\ 0 & 0 & 0 & 0.2z_1 & -0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.002(1+\delta_1) \end{bmatrix}.$$

Since this system has full relative degree there is not an internal dynamics. Therefore, the stability of its DAR (2.16) results in the stability of whole system.

Actuation Limit on New Input Control

Notice that in order to have a fair comparison with the similar research study in (Rohr et al., 2009) it is required to consider a control limit condition for the new input $v(t)$. Denoting the maximum available control norm as v_{max}, it follows that:

$$||v||^2 \leq v_{max}^2, v = Kz \Rightarrow (Kz)^T(Kz) \leq v_{max}^2,$$

with $K = YQ^{-1}$, according to Theorem 4.2, and $Q = Q^T$ we obtain $z^TQ^{-1}Y^TYQ^{-1}z \leq v_{max}^2$ or equivalently

$$\frac{1}{v_{max}^2}z^T[Q^{-1}Y^TYQ^{-1}]z \leq 1. \tag{5.1}$$

Since we are investing initial conditions and invariant ellipsoidal region satisfying $z^TQ^{-1}z \leq 1$ we can ensure that (5.1) is satisfied by imposing (sufficient condition):

$$\frac{1}{v_{max}^2}z^T[Q^{-1}Y^TYQ^{-1}]z \leq z^TQ^{-1}z \leq 1.$$

So, we have

$$\frac{1}{v_{max}^2}Q^{-1}Y^TYQ^{-1} \leq Q^{-1}.$$

Multiplying both sides by Q on the right and left we obtain

$$\frac{1}{v_{max}^2}Y^TY \leq Q \Leftrightarrow Q - \frac{1}{v_{max}^2}Y^TY \geq 0.$$

Therefore, by taking the Shur complement of the above inequality we achieve the following LMI which imposes a saturation limit on the control input:

$$\begin{bmatrix} Q & Y^T \\ Y & v_{max}^2I \end{bmatrix} \geq 0. \tag{5.2}$$

Remark 5.1. Since we are taking the I/O feedback linearization there exists a nonlinear relation between the real input u and virtual input v based on equation (2.14). Therefore, computing v_{max} from u_{max} is not in general an easy task.

Now regarding Theorem 4.2 the SDP problem (4.10) can be solved for the inverted pendulum system by including the LMI constraint in (5.2). For the sake of comparison the same physical parameters of the inverted pendulum used in (Rohr et al., 2009) are also considered here as presented in Table 5.1.

TABLE 5.1: Parameters of Inverted Pendulum

Parameter	Value
M_0	2 Kg
l	1 m
g	$9.8 \frac{m}{s^2}$
b_0	$0.5 \frac{N.s}{m}$

The same maximum control used in (Rohr et al., 2009) is also considered here. For the given control $K = [-1 \ -2]$ and the associated DOA one can compute the exact $v_{max} = K||x||_{max}$ from (Rohr et al., 2009). In this case, $v_{max} = 0.25$ and it can be used in LMI (5.2) as a given parameter. In this regard, to find a feasible solution of SDP problem (4.10) which satisfies stability condition of Theorem 4.2, the following bounds on states polytope and uncertain parameters are obtained:

$$|z_1| \leq 0.17, \ \ |z_2| \leq 0.18,$$
$$|\delta_1| \leq 0.105, \ \ |\delta_2| \leq 1,$$

together with the following estimate of DOA and synthesized control matrix K:

$$\Omega_z = \left\{ z \in \mathbb{R}^2 | z^T \begin{bmatrix} 47.99 & 23.95 \\ 23.95 & 42.81 \end{bmatrix} z \leq 1 \right\}, K = \begin{bmatrix} -1.1928 & -1.6022 \end{bmatrix}.$$

Figure 5.3 depicts the estimated DOA Ω_z, together with the system trajectories, denoted by solid curves, with all possible bounds of uncertain parameters and the estimated DOA, denoted by the dotted ellipsoid which is obtained in (Rohr et al., 2009). As it can be seen, the ellipsoidal DOA obtained by synthesis problem is larger than that of (Rohr et al., 2009) which is obtained by a priori synthesized control matrix K. To investigate the robustness of the closed-loop system the uncertainty parameter δ_1 is gradually increased to evaluate its effect on the DOA volume. As depicted in Figure 5.4 there exists a sudden fall of DOA volume once $\delta_1 > 0.465$.

Table 5.2 shows the comparative results obtained by the proposed control design and (Rohr et al., 2009). This table shows that although similar results are obtained, a moderately larger volume of DOA is estimated.

TABLE 5.2: Comparison of Results of DOA for Inverted Pendulum

	Proposed approach	(Rohr et al., 2009)
States Polytope	$|x_1| \leq 0.17, |x_2| \leq 0.18$	$|x_1| \leq 0.15, |x_2| \leq 0.15$
Uncertainty Polytope	$|\delta_1| \leq 0.105, |\delta_2| \leq 1$	$|\delta_1| \leq 0.097, |\delta_2| \leq 0.99$
log(det(Q))	-7.3	-7.96
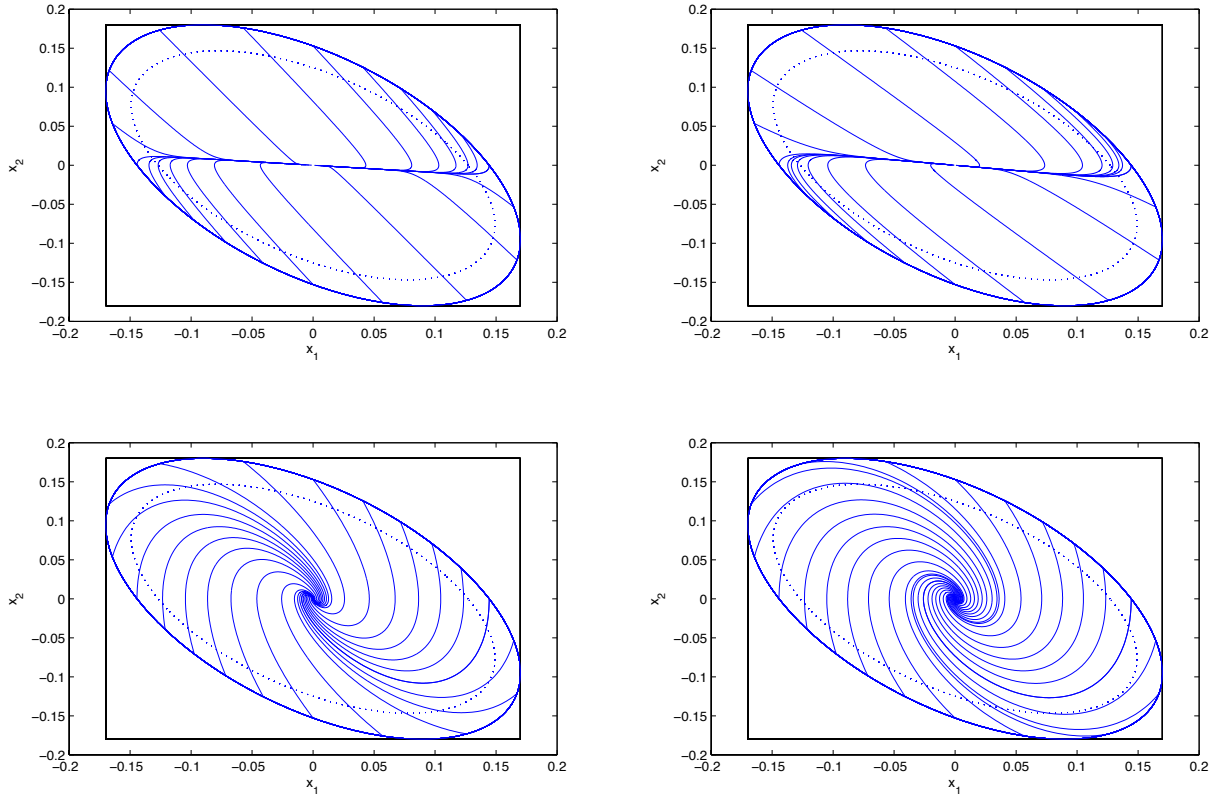
FIGURE 5.3: Estimated ellipsoidal DOA and Phase Trajectories of States with Different Bounds of δ_1 and δ_2.

5.3   Lorenz System

To illustrate the effectiveness of our approach in finding guaranteed DOA for nonlinear system (2.29) in DAR form, an analysis problem is considered in this section. The following system describes a Lorenz attractor and it is borrowed from (Valmórbida et al., 2010):

$$
\begin{aligned}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2, \\
\dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3, \\
\dot{x}_3 &= -b x_3 + x_1 x_2,
\end{aligned}
\tag{5.3}
$$
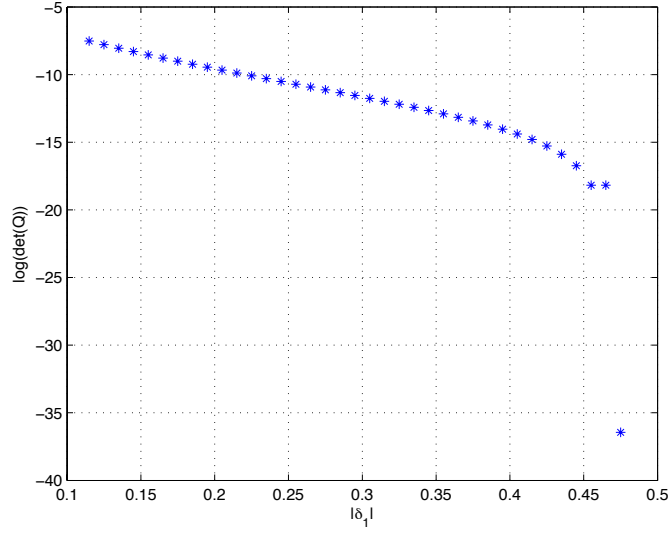
FIGURE 5.4: variation of DOA volume versus uncertainty $|\delta_1|$ growth.

where σ, ρ and b are positive real scalars. Following the same approach in (Valmórbida et al., 2010) the error dynamics of the system around the equilibrium point $x_{ep} = [\sqrt{b(\rho-1)} \ \sqrt{b(\rho-1)} \ \rho-1]^T$, for the values $\sigma = 10, \rho = 4$ and $b = \frac{8}{3}$, is given by:

$$
\begin{aligned}
\dot{X}_1 &= -10X_1 + 10X_2, \\
\dot{X}_2 &= X_1 - X_2 - \sqrt{8}X_3 - X_1X_3, \\
\dot{X}_3 &= \sqrt{8}X_1 + \sqrt{8}X_2 - \frac{8}{3}X_3 + X_1X_3,
\end{aligned}
\tag{5.4}
$$

in which $X = x - x_{ep}$. Hence, system (5.4) can be recast as DAR (2.29) such that:

$$
\pi(X) = \begin{bmatrix} 0 & -X_1X_3 & X_1X_2 \end{bmatrix}^T, \ A_1(X) = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{8} \\ \sqrt{8} & \sqrt{8} & -\frac{8}{3} \end{bmatrix}, \ A_2(X) = I_3,
$$

$$
\Pi_1(X) = \begin{bmatrix} 0 & 0 & 0 \\ X_3 & 0 & 0 \\ -X_2 & 0 & 0 \end{bmatrix}, \Pi_2(X) = I_3.
$$

The SDP problem (4.26) can be solved for different values of γ (the constant scalar in LMIs (4.18)) , as shown in Figure 5.5, such that the LMI (4.18) remains feasible. For illustration, if one pick $\gamma = 10^{-4}$ the following estimation of the largest guaranteed ellipsoidal DOA is obtained:
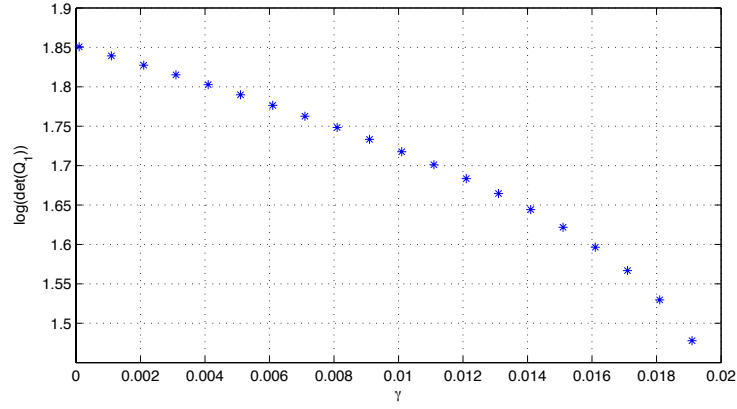
FIGURE 5.5: Variation of the largest guaranteed ellipsoidal DOA volume for different values of γ.

$$\Omega(P,1) = \left\{ X \in \mathbb{R}^3 | X^T \begin{bmatrix} 0.2112 & 0.0893 & 0.0386 \\ 0.0893 & 1.3341 & -0.4532 \\ 0.0386 & -0.4532 & 0.7510 \end{bmatrix} X \leq 1 \right\},$$

with the corresponding state space polytopic set $\mathbb{X}$ defined by:

$$|X_1| \leq 2.3, \ |X_2| \leq 1, \ |X_3| \leq 1.32.$$

Therefore, the proposed approach estimates the volume of $\Omega(P,1)$ for system (5.4) with $\log(\det(Q_1)) = 1.85$ which is larger than the estimated volume obtained by (Valmórbida et al., 2010) in which $\log(\det(Q_1)) = 0.3691$. The estimated DOA for the error dynamics (5.4) together with some trajectories, initiating outside of it, are depicted in Figure 5.6.

According to the Remark 2.1 in order to check the effect of different DARs in the stability analysis results we solve the SDP problem (4.26), with the same γ, for the following different DAR of system (5.4):

$$\pi(X) = \begin{bmatrix} 0 & -X_3 & X_2 \end{bmatrix}^T, \ A_1(X) = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{8} \\ \sqrt{8} & \sqrt{8} & -\frac{8}{3} \end{bmatrix}, \ A_2(X) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X_1 & 0 \\ 0 & 0 & X_1 \end{bmatrix},$$

$$\Pi_1(X) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \Pi_2(X) = I_3.$$
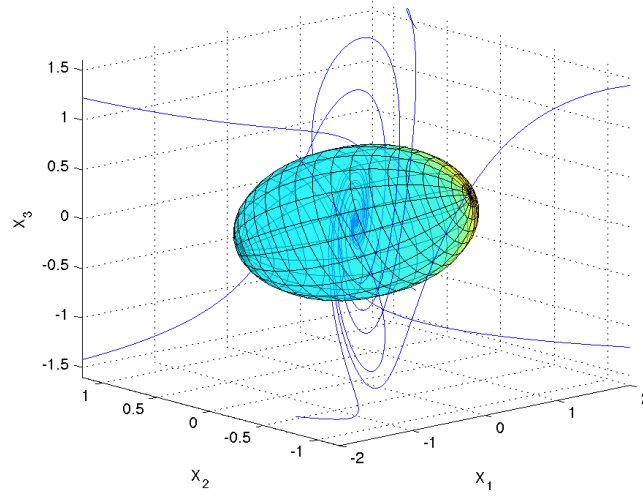
FIGURE 5.6: Guaranteed DOA for the Lorenz system together with some state trajectories.

Therefore, the estimation of maximum DOA with the new DAR is calculated as:

$$\Omega(P,1) = \left\{ X \in \mathbb{R}^3 | X^T \begin{bmatrix} 1.0530 & -0.7145 & -0.2332 \\ -0.7145 & 1.1504 & 0.0545 \\ -0.2332 & 0.0545 & 0.4040 \end{bmatrix} X \leq 1 \right\},$$

with the polytopic set of state-space $\mathbb{X}$ characterized by:

$$|X_1| \leq 1.4, \ \ |X_2| \leq 1.3, \ \ |X_3| \leq 1.8.$$

In conclusion, the new estimation of DOA has the volume $\log(\det(Q_1)) = 1.44$ which shows that with a different DAR more conservative stability result is obtained.

5.4   SISO System with Input Saturation

The following case is an illustrative example, without parametric uncertainty, that was employed in (Oliveira et al., 2012), where the circle criterion and a generalized sector bound condition were used in the theoretical development. The following nonlinear system with saturated input is considered:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= (1 + x_1^2)x_1 + (2 + 8x_2^2)x_2 + \text{sat}(u(t)),
\end{aligned}
\tag{5.5}
$$

whose DAR (2.29) can be obtained as

$$
\pi = \begin{bmatrix} x_1^2 & x_2^2 \end{bmatrix}^T, A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ x_1 & 8x_2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$

$$
\Pi_1 = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}, \Pi_2 = -I_2, \Pi_3 = 0_{2\times 1}.
$$

Considering the same saturation bound of $u_0 = 1.96$ used in (Oliveira et al., 2012), the SDP (4.26) is solved for different values of γ to obtain the maximum possible volume for the guaranteed ellipsoidal DOA. Figure 5.7 illustrates that this maximum is attained with $\gamma = 0.05$ for which $\log\{det(Q_1)\} = -2.26$. For values of $\gamma < 0.05$ the volume $\log\{det(Q_1)\}$ remains approximately constant. In addition, for $\gamma = 0.05$ the corresponding ellipsoidal DOA $\Omega(P, 1)$ lies inside the state polytope described by:

$$
|x_1| \le 0.74, \quad |x_2| \le 0.46.
$$

Accordingly, the estimated ellipsoidal DOA (depicted in Figure 5.8 by solid lines) and synthesized control gain K are obtained as:

$$
\Omega(P, 1) = \left\{ x \in \mathbb{R}^2 | x^T \begin{bmatrix} 2.0331 & 1.0436 \\ 1.0436 & 5.2616 \end{bmatrix} x \le 1 \right\}, \quad K = \begin{bmatrix} -9.2093 & -43.0533 \end{bmatrix}.
$$

Figure 5.8 shows the corresponding guaranteed ellipsoidal DOA together with state trajectories for different initial conditions in solid line. Also, Figure 5.8 depicts the guaranteed ellipsoidal DOA in dotted line obtained, with the same saturation limit, using the approach in (Oliveira et al., 2012). Accordingly, the estimated volume of the guaranteed ellipsoidal DOA obtained
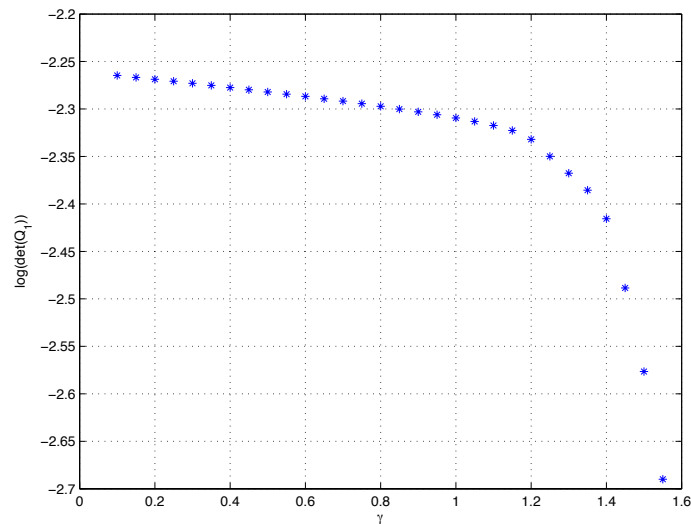
FIGURE 5.7: The variation of largest guaranteed ellipsoidal DOA volume for different values of γ.
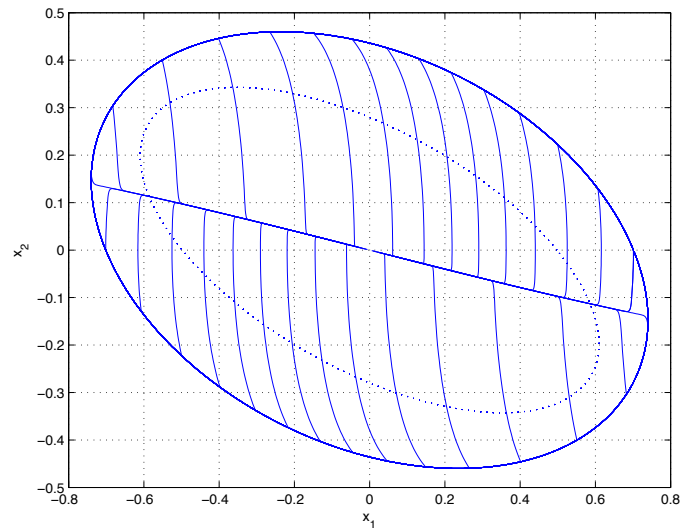
FIGURE 5.8: DOA and states trajectories for system (5.5).

TABLE 5.3: Characteristics of DOA of system (5.4).

	(Oliveira et al., 2012)	Proposed approach
States Polytope	$\lvert x_1 \rvert \leq 0.61, \lvert x_2 \rvert \leq 0.35$	$\lvert x_1 \rvert \leq 0.74, \lvert x_2 \rvert \leq 0.46$
$\log(\det(Q_1))$	-3.5375	-2.2626

with the proposed approach is greater than that of (Oliveira et al., 2012) and encompasses their estimated DOA. In this context Table 5.3 shows some comparative results as well.

5.5 MIMO System with Input Saturation

A MIMO nonlinear system is adapted from (Da Silva et al., 2014) for the sake of further evaluation of the theoretical development. The system's model is:

$$\dot{x_1} = (x_1 + 2(1 + \delta))x_1^2 + 10x_2 + 10\text{sat}(u_1(t)),$$
$$\dot{x_2} = -100x_1 - 30x_2 + 10\text{sat}(u_2(t)),$$
(5.6)

whose DAR (2.29) is formed by:

$$A_1 = \begin{bmatrix} 0 & 10 \\ -100 & -30 \end{bmatrix}, \ A_2 = \begin{bmatrix} x_1 + 2(1 + \delta) \\ 0 \end{bmatrix}, \ A_3 = 10I_2,$$

$$\pi = x_1^2, \ \Pi_1 = \begin{bmatrix} x_1 & 0 \end{bmatrix}, \ \Pi_2 = -1, \ \Pi_3 = 0_{1\times2}.$$

Note that the uncertainty δ does not exist in (Da Silva et al., 2014). Therefore, for the sake of fair comparison, first, we assume that $\delta = 0$. Considering the same saturation bound of $u_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ used in (Da Silva et al., 2014) SDP (4.26) is solved for different values of γ to search for the maximum possible volume for the guaranteed ellipsoidal DOA. Figure 5.9 shows that greater volumes can be obtained as γ gets closer to zero. However, for some γ in the vicinity of zero the LMI (4.18) does not have feasible solutions. A small enough γ for which LMI (4.18) is feasible is $\gamma = 5 \times 10^{-4}$, and the corresponding volume $\log(\det(Q_1)) = 7.4$. Moreover, the corresponding DOA lies inside the set $\mathbb{X}$ defined by:
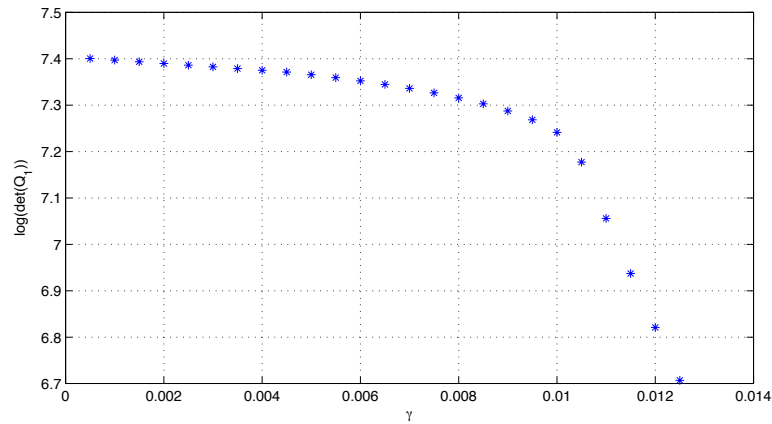
FIGURE 5.9: Variation of the largest guaranteed ellipsoidal DOA volume for different values of γ.

$$|x_1| \leq 4.$$

Therefore, the estimated DOA together with the synthesized control gain are given by:

$$\Omega(P,1) = \left\{ x \in \mathbb{R}^2 | x^T \begin{bmatrix} 0.1374 & 0.0271 \\ 0.0271 & 0.0098 \end{bmatrix} x \leq 1 \right\}, \quad K = \begin{bmatrix} -332.3525 & -65.4635 \\ -820.8413 & -296.3322 \end{bmatrix}.$$

The DOA, obtained above, is depicted in Figure 5.10 with different system trajectories initiating within its border. The results are compared with those of (Da Silva et al., 2014) (obtained from static anti-windup control approach). Since the DOA in (Da Silva et al., 2014) is obtained in the augmented state-space, after they defined new dynamics control variables, their DOA volume is calculated in the original ($x_1 - x_2$) state-space plane. As shown in Table 5.4 more relaxed bounds of variations for the state-space polytopic set and larger volume in $x_1 - x_2$ plane are obtained for u_0.

TABLE 5.4: Characteristics of state space polytope of system (5.6).

	(Da Silva et al., 2014)	Proposed approach
States Polytope	$|x_1| \leq 3$	$|x_1| \leq 4$
$\log(\det(Q_1))$	5.65 (corresponding area of the projected DOA on $x_1 - x_2$ plane)	7.4

Now, we consider the uncertainty in the system and investigate its variation on the volume of DOA with the same actuation and state bounds. Figure 5.11 shows the effect of uncertainty
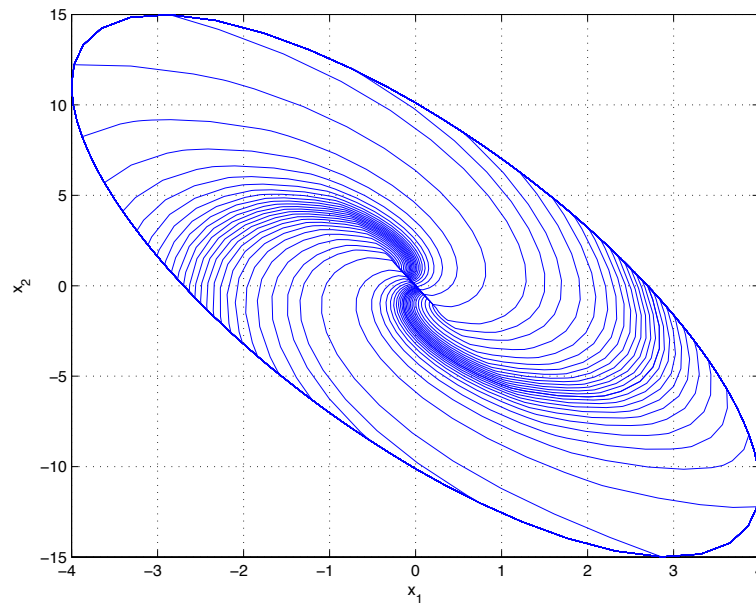
FIGURE 5.10: Guaranteed ellipsoidal DOA with the system trajectories.
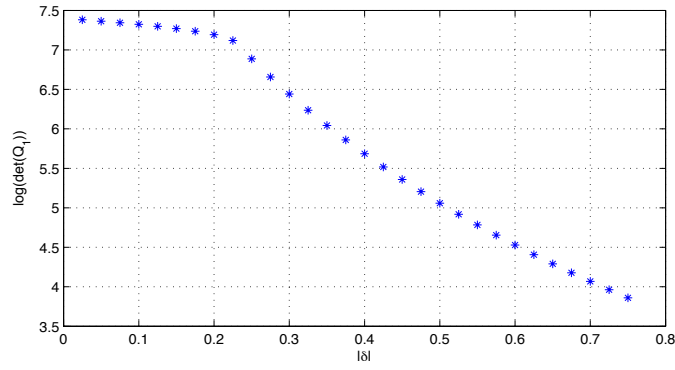
FIGURE 5.11: Variation of DOA volume versus the growth of uncertainty bound.
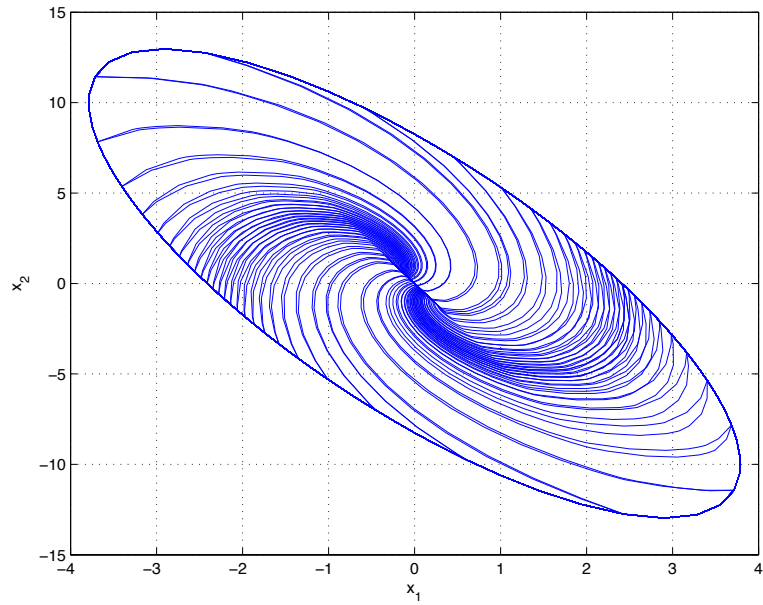
FIGURE 5.12: Guaranteed ellipsoidal DOA with the system trajectories in the presence of uncertainty.

bound growth on the estimation of maximum guaranteed ellipsoidal DOA. Accordingly, the volume of DOA decreases moderately versus the uncertainty bounds growth while $|\delta| \leq 0.225$. However, for $|\delta| \geq 0.25$ the volume decreases with higher pace as the uncertainty bounds increase. The DOA with the system trajectories for $|\delta| = 0.25$ is depicted in Figure 5.12.

5.6 Feedback Linearized Pendulum with Input Saturation

In this example the same feedback linearized inverted pendulum model described in (2.22) is studied with some differences such that, firstly, since the mapping does not change the system's

coordinates between (2.19) and (2.22) ($z = [x_1 \ x_2]^T$) and (2.19) is input-state linearizable (there is no internal dynamics) (2.22) is rewritten in x-coordinates; and secondly, we consider the new control input $v(t) = \text{sat}(u(t))$ to be an input saturation control. That is because we can check whether or not assuming input-saturated system can estimate larger volume of DOA. If so, the advantage of considering saturated input to the system would unfold. Therefore, we rewrite (2.22) in x-coordinates and put $v(t) = \text{sat}(u(t))$ as following:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\frac{b_0 \delta_2}{M_0(1+\delta_1)}x_2 + \frac{\delta_1}{1+\delta_1}\left(\frac{2x_1 x_2^2}{1+x_1^2} + \frac{g}{l}x_1\right) + \frac{1}{1+\delta_1}\text{sat}(u(t)).
\end{aligned}
\tag{5.7}
$$

Therefore, the DAR (2.29) of above input-saturated system can be obtained with:

$$
\pi(x, \delta p, \text{sat}(u(t))) = \begin{bmatrix} \frac{x_1}{1+\delta_1} & \frac{x_2}{1+\delta_1} & \frac{x_1 x_2 \delta_1}{(1+\delta_1)(1+x_1^2)} & \frac{x_1 \delta_1}{1+x_1^2} & \frac{x_1^2 \delta_1}{1+x_1^2} & \frac{\text{sat}(u(t))}{1+\delta_1} \end{bmatrix}^T,
$$

$$
A_1(x, \delta p) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad
A_2(x, \delta p) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{g}{l}\delta_1 & -\frac{b}{M}\delta_2 & 2x_2 & 0 & 0 & -\delta_1 \end{bmatrix}, \quad
A_3(x, \delta p) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$

$$
\Pi_1(x, \delta p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \delta_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad
\Pi_2(x, \delta p) = \begin{bmatrix} -1-\delta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-\delta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-\delta_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -x_1 & 0 \\ 0 & 0 & 0 & x_1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-\delta_1 \end{bmatrix},
$$

$$
\Pi_3(x, \delta p) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T.
$$

Now regarding Theorem 4.7, the SDP problem (4.26) is solved for the DAR of system (5.7) (with $u(t) = Kx(t)$) which can be recast as (4.13) with $m = 1$ and $D_s, D_s^- \in \{0, 1\}$, $s = 1, 2$. The physical parameters are given in Table 5.1 and the saturation limit $u_0 = 0.25$. The approach proposed in (Oliveira et al., 2012), which is based on the generalized sector bound condition and the circle criterion, is also implemented on this example for further comparisons.

The largest guaranteed ellipsoidal DOA, with maximum volume, can be obtained for $\gamma = 0.001$ when solving (4.26). Figure 5.13 depicts the variation of DOA volume for several values of γ. For $\gamma = 0.001$, $\log\{\det(Q_1)\} = -6.8$, for which the corresponding DOA $\Omega(P, 1)$ lies inside the state polytope defined by:
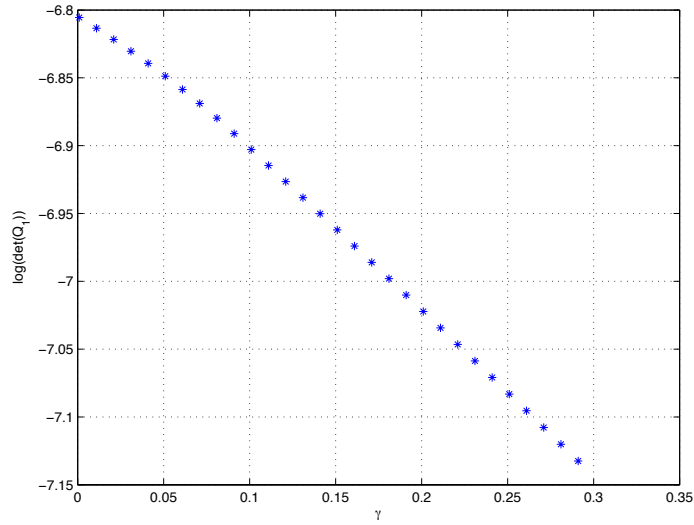
FIGURE 5.13: Variation of the largest guaranteed ellipsoidal DOA volume for different values
of γ.

$$|x_1| \leq 0.19, \quad |x_2| \leq 0.22,$$

and the bounds for the uncertainties:

$$|\delta_1| \leq 0.1, \quad |\delta_2| \leq 0.99,$$

whose amplitudes still ensure the asymptotic stability of system trajectories inside $\Omega(P, 1)$. In this case, the estimated guaranteed ellipsoidal DOA (which is also depicted in Figure 5.14) and the control gain K are given by:

$$\Omega(P, 1) = \left\{ x \in \mathbb{R}^2 | x^T \begin{bmatrix} 43.3560 & 22.5876 \\ 22.5876 & 32.5898 \end{bmatrix} x \leq 1 \right\}, \quad K = \begin{bmatrix} -84.5238 & -132.0494 \end{bmatrix}.$$

Figure 5.14 illustrates the asymptotic convergence of states inside DOA for different initial conditions. Also, a sensitivity analysis is performed to evaluate the robustness of linear state feedback saturated control by gradually increasing the bound of parametric uncertainty δ_1 until some instability occurs. Accordingly, as depicted in Figure 5.15, unstable behavior is detected for some trajectories of the closed-loop system when the bound for the norm of δ_1 is greater than 0.137, which is well far away from the originally considered bound of 0.1.
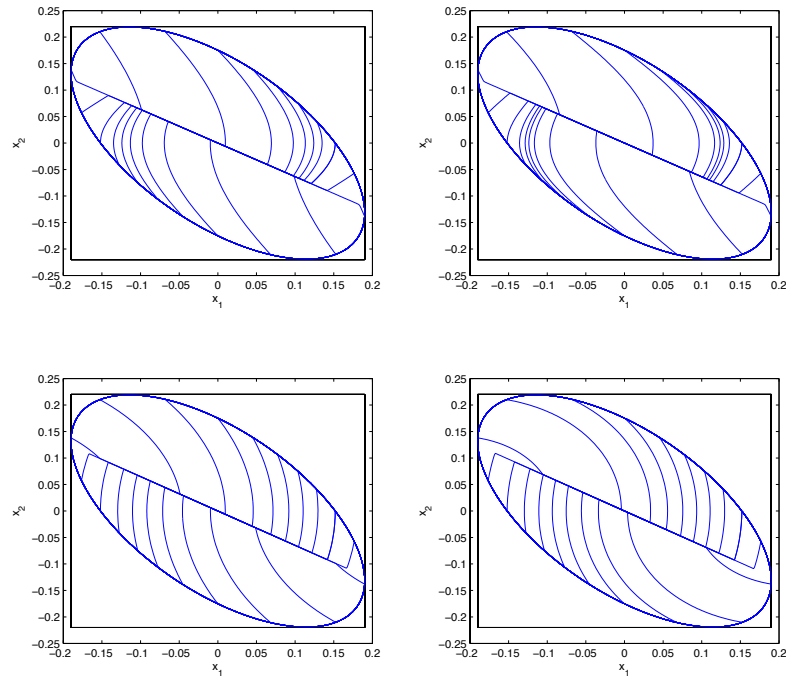
FIGURE 5.14: Estimated DOA and states trajectories of the inverted pendulum with different bounds of δ_1 and δ_2.
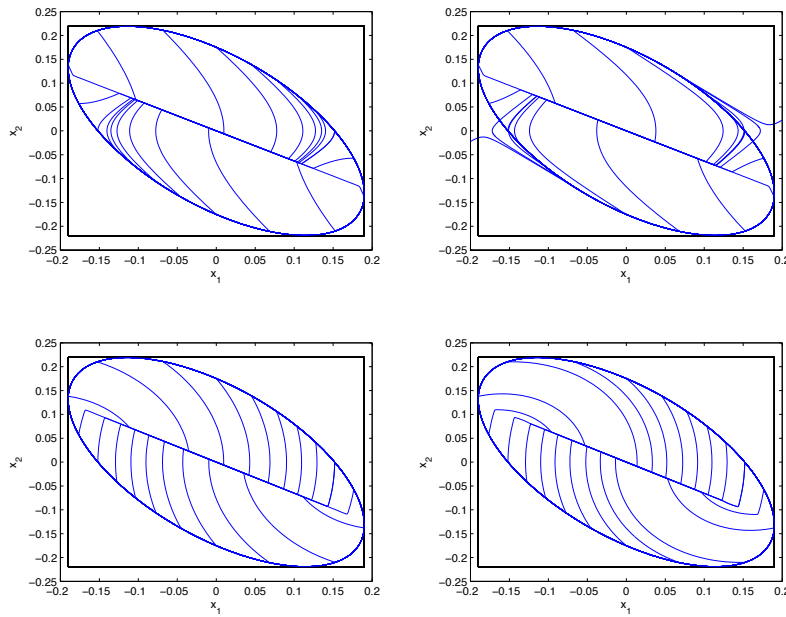
FIGURE 5.15: States trajectories of the inverted pendulum and DOA for $|\delta_1| \geq 0.137$ and different bounds of δ_2.

TABLE 5.5: Characteristics of robustly asymptotically stable DOA of Inverted Pendulum.

	(Rohr et al., 2009)	(Oliveira et al., 2012)	Proposed approach
States Polytope	$\|x_1\| \leq 0.15$ $\|x_2\| \leq 0.15$	$\|x_1\| \leq 0.19$ $\|x_2\| \leq 0.21$	$\|x_1\| \leq 0.19$ $\|x_2\| \leq 0.22$
Uncertainty Polytope	$\|\delta_1\| \leq 0.097$ $\|\delta_2\| \leq 0.99$	$\|\delta_1\| \leq 0.1$ $\|\delta_2\| \leq 0.99$	$\|\delta_1\| \leq 0.1$ $\|\delta_2\| \leq 0.99$
$\log(\det(Q_1))$	-7.965	-7.11	-6.80

Table 5.5 shows the comparison of our study with both the results of (Rohr et al., 2009) and the results generated by applying the circle criterion-based LMIs in (Oliveira et al., 2012). Compared with (Rohr et al., 2009), a larger guaranteed ellipsoidal DOA is estimated while its robustness against uncertain parameter δ_1 is greater. Moreover, compared with the approach in (Oliveira et al., 2012), although the same bounds of uncertainties were employed, our method estimates a larger guaranteed ellipsoidal DOA.

Chapter 6

Final Remarks

6.1 Overview

This study, tackled with the problem of robust stability of uncertain nonlinear systems after applying feedback linearization strategy. The parametric uncertainties were assumed to be bounded and their bound were known. In this circumstance, the study relied on systems with rational vector fields with respect to the uncertainties and system states. Hence, the DAR tool was used to identically represent the approximately I/O linearized system in such a way that one can use LMI based stability analysis and control synthesis problem. The outcome, for the systems with total relative degree equal to the states dimension, was the synthesis of a linear state feedback controller in new coordinates through solving an SDP problem. The SDP problem found a control gain which estimated the maximum ellipsoidal DOA inside the polytope of system states in new coordinates. The approach was examined by an illustrative example, an inverted pendulum, and the comparison with the recent reference in Rohr et al. (2009) illustrated its efficiency in terms of achieving lager estimation of DOA volume. However, for I/O feedback linearizable systems, it was required to investigate the stability of internal dynamics after obtaining an approximate linearization if the total relative degree of the system was smaller than states dimension. In this respect, the same sufficient LMIs for I/O linearized systems in the form of DAR, were used under the condition of input-to-state stability of internal dynamics and the corresponding SDP problem was solved subject to those LMIs in order to estimate the maximum hyper ellipsoidal DOA in new coordinates. It was shown that if such optimization problem is feasible the input-to-state stability of internal dynamics leads to the regional asymptotic stability of the whole system. The outcome, for a borrowed example, owning internal dynamics, from

a previous study, showed that the full states trajectories are asymptotically stabilizable within the DOA of output dynamics and the pre-determined bounds of internal dynamics state in the presence of uncertainties.

In another investigation, the regional robust stabilization problem of input-saturated uncertain nonlinear systems was investigated in this work with emphasis on the maximization of the associate guaranteed ellipsoidal DOA volume. The class of uncertain MIMO nonlinear systems under consideration was representable in a DAR form, with polytopic parametric uncertainties, and whose matrices are affine on the states and parameters. In addition, the ellipsoidal DOAs are searched for a priori defined polytopic regions of the state space. A part of mathematical development relied on the main idea of representing the saturated input as a convex combination of an unsaturated input and properly chosen vectors (Hu and Lin, 2001), together with the simultaneous use of Finsler Lemma and linear annihilators (Trofino and Dezuo, 2014). The numerical experiments support the conjecture that the derived LMI conditions for control synthesis are less conservative.

6.2   Uncertain Nonlinear Systems Representations

As comprehensively explained in chapter 2, some representations of uncertain nonlinear systems were addressed and compared. Overall, the DAR representation were an intriguing tool in the context of regional robust stability and in terms of its simpleness over LFR representation when it is applied to systems with higher degree of nonlinearities, although these two representations are closely related conceptually. However, these representations are confined to the class of systems with rational vector fields with respect to the states and uncertainties.

The inverse dynamics technique for I/O linearizable rational systems owning uncertainties were applied to approximately linearize such class of systems around instantaneous operating points. Thereafter, the remaining quasi-canonical form could still remain rational such that it was representable in DAR form. This characteristic enabled us to take into account the information of remaining nonlinearities in the system caused by inexact feedback linearization attempt. Therefore, regardless of internal dynamics at first, the DAR of quasi canonical dynamics were obtained to facilitate robust stability analysis and control synthesis for the new input control.

Despite applying DAR for the control-affine systems, this representation was also applicable for systems with saturation input which are closer to their real dynamical models due to actuation

restrictions. In this context, the combination of DAR form, as a beneficial tool, and polytopic representation of input saturation was a constructive preliminary for the stability analysis purpose and the outcomes showed proficiency of such representations.

6.3 Stability Analysis

Chapter 3 dealt with the notion of quadratic stability conditions and state-space polytopic description of uncertain nonlinear systems. Within this context, the set of system states and corresponding DOAs, in terms of quadratic Lyapunov candidate functions, were defined and it was shown that how one can propose sufficient LMI conditions such that the DOAs are always inside the system's state-space polytopic sets. Further, the polytopic description of matrices in DAR systems were presented in order to enable proposition of stabilizing LMI conditions in Chapter 4 for such representations either for the approximately I/O linearized systems or for the input saturation systems.

6.4 State Feedback Control Synthesis

Chapter 4 tackled the problem of designing a static state feedback control for the DAR of approximate I/O linearized systems and the DAR of uncertain nonlinear systems with input saturation. In both cases the static feedback control gains, as decision variables, and the associate DOA were the outcomes of proposed sufficient stabilizing LMI conditions. The obtained static feedback gains guaranteed asymptotic convergence of every systems' trajectory initiating inside the DOA in the presence of parametric uncertainties whose variations were confined to compact polytopic sets.

6.5 Possible Future Work

6.5.1 Reducing Conservatism

For the class of input saturation systems the possible future investigations could be performed at least in two areas. First, one can try to reduce the number of sufficient stabilizing LMIs, since the number of LMIs to be satisfied grows exponentially with the number of states, parameters

and control inputs, which can prevent the search for numerical solutions in practice. Second, it could be possible to develop less conservative stabilizing LMIs by choosing more wisely the block matrix-variable $\mathcal{N}$ in Finsler Lemma, such that the number of decision variables increases.

6.5.2 Study of Non-minimum Phase Systems

Future studies could investigate the problem of non-minimum phase systems which have unstable internal dynamics. In this regard many investigations have been performed for this class of systems. The studied methodologies are classified in Rajput and Weiguo (2014) as followings:

- Approximate feedback linearization: In this method a slightly non-minimum phase nonlinear system is transformed into an approximate minimum phase one.

- Output redefinition: The method relies on finding a new system output in which the internal dynamics states explicitly appear in the stability analysis.

- Real-Zero elimination: This method tries to approximate a class of non-minimum phase systems by minimum phase ones.

- Stable inversion: In this method a bounded solution for input control and system states is computed such that it imposes the requirement for the inverse solution to be stable.

- Output-feedback stabilization: This method is used for nonlinear non-minimum phase systems whose some of states are not measurable in order to be available for output feedback.

6.5.3 Inverse Dynamics in The Context of Model Predictive Control

Inverse dynamics control can be used in the context of trying to find the appropriate inputs that will drive the nonlinear uncertain dynamical system to a prescribed reference trajectory y_{ref}, as it is depicted in Figure 6.1. In this case, the use of inverse dynamics is closer to the Optimal Control approach (Kirk, 2004) as applied to Nonlinear Model Predictive Control (NMPC) methods (García et al., 1989, Allgower and Zheng, 2000, Diehl et al., 2009), particularly when the instantaneous values of the inputs are calculated as a result of some underlying optimization procedure (e.g. to minimize a quadratic energy-like function over a finite or infinite horizon, at each time step). The robust LTI controller, in this case, can be used to guarantee the overall

system stability and performance around the reference trajectory. Assuming that the inverse dynamics is only an approximation since the system model is uncertain, incremental stabilizing corrections should be added by the robust LTI controller to the resulting inputs used to drive the nonlinear system.

6.5.4 Inverse Dynamics in The Context of Passivity Theory

Other possible future study can be investigated within the context of passivity theory where the approximate inverse dynamics of an uncertain nonlinear system, which was calculated in Section 2.4.1 of Chapter 2 throughout the nominal parts of system's vector fields, would be considered as an input nonlinearity such that in closed-loop form it satisfies passivity condition. More precisely, consider again the uncertain nonlinear system (2.4), in the closed-loop scheme of Figure 6.1, which is representable in DAR form (2.7). Now, if the approximate inverse model, in red block of Figure 6.1, can be calculated from (2.14) as a nonlinear input $\psi(x) = -G_*^{-1} f_*$ together with the robust LTI controller, in blue block, to be $\bar{u}(t) = Ky(t)$, where $K \in \mathbb{R}^{m \times m}$, with $y_{ref} = 0$, the system in closed-loop form, with the input control $u(t) = \bar{u}(t) + \psi(x)$, can be modeled as:

$$\dot{x}(t) = A_1(x, \delta)x + A_2(x, \delta)\pi + A_3(x, \delta)(\bar{u} + \psi(x)), \qquad (6.1)$$
$$0 = \Pi_1(x, \delta)x + \Pi_2(x, \delta)\pi + \Pi_3(x, \delta)(\bar{u} + \psi(x)),$$
$$y = h(x).$$

Now the uncertain nonlinear system (6.1) is subject to the input nonlinearity $\psi(x)$ which is derived from the approximate I/O linearization attempt. The system structure encourages applying
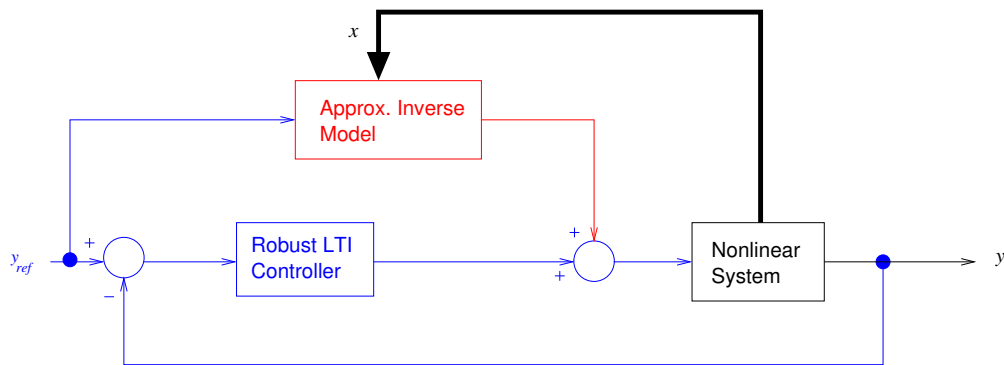
FIGURE 6.1: Inverse Dynamics based Control in the context of nonlinear model predictive control or passivity theory.

passivity theory which might lead to the whole system stability including internal dynamics. To clarify the problem, according to the notion of passivity, the system (6.1) is output strictly passive if there exists a continuously differentiable positive semidefinite function $V(x)$ such that $\bar{u}^T y \geq \dot{V} + y^T \psi$ and $y^T \psi \geq 0$, $\forall y \neq 0$ Khalil (2002). This can be interpreted within the context of supplied energy and storage energy in the system. That is if we consider the flow of energy into the system as supply function $S(x, t) = \int_t^T (\bar{u}^T y - y^T \psi) \, dt$, $y^T \psi \geq 0$, and the total storage energy in the system to be $V(x)$, one can imply that the system is output strictly passive if $S(x, t) \geq V(x)$ for all $t \geq 0$ and $y \neq 0$.

Therefore, one can derive sufficient LMIs and seek their feasibility based on the notion of passivity to investigate stability of the system.

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