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Brief Paper

Robust filtering with guaranteed energy-to-peak performance — an $\mathscr{LMI}$ approach☆

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Abstract

The problem of robust energy-to-peak filtering for linear systems with convex bounded uncertainties is investigated in this paper. The main purpose is to design a full order stable linear filter that minimizes the worst-case peak value of the filtering error output signal with respect to all bounded energy inputs, in such a way that the filtering error system remains quadratically stable. Necessary and sufficient conditions are formulated in terms of linear Matrix Inequalities — $\mathscr{LMI}$s, for both continuous- and discrete-time cases. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Since the Kalman filtering theory has been introduced (Kalman, 1960) much effort has been devoted to the problem of estimating an output error signal (linear combination of the states) through a filter structure such that a guaranteed performance criteria is minimized in an estimation error sense. In this setting, the $\mathscr{H}_2$ filtering design arises as an efficient strategy whenever the noise input is assumed to have a known power spectral density. In the literature, the $\mathscr{H}_2$ filtering problem has been faced using Riccati-based approaches (Shaked & de Souza, 1994; Xie & de Souza, 1995), and more recently by means of linear matrix inequalities ($\mathscr{LMI}$s) (Khargonekar, Rotea & Baeyens, 1996; Palhares & Peres, 1998).

In the case where there exists insufficient statistical information about the noise input, two strategies can be employed. The first one is the well-known $\mathscr{H}_\infty$ filtering design, in which the input is supposed to be an energy signal and the energy-to-energy gain is minimized, or simply bounded by a prescribed value. Many papers have dealt with $\mathscr{H}_\infty$ filter design (as well as with the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ filtering problem), even when uncertain parameters are taken into account. Basically, the solutions are obtained through Riccati-like equation (Takaba & Katayama, 1996; Park & Kailath, 1997; Xie & de Souza, 1995), or $\mathscr{LMI}$s (Khargonekar et al., 1996; Li & Fu, 1997; Palhares & Peres, 1998; Geromel, Bernussou, Garcia & de Oliveira, 1998; Geromel & de Oliveira, 1998; Palhares & Peres, 1999).

The second approach is the peak-to-peak filtering design, that is, the problem of finding a linear filter which minimizes the worst-case peak value of the filtering error for all bounded peak values of the input signals (Nagpal, Abedor & Poolla, 1996; Vincent, Abedor, Nagpal & Khargonekar, 1996; Voulgaris, 1996). In other words, the maximal peak-to-peak gain of the filtering error system is used as performance criterion (ℓ_1 norm for discrete-time, $\mathscr{L}_1$ norm for continuous-time systems). In the uncertain case, an algorithm for the guaranteed ℓ_1 norm computation has been proposed in Fialho and Georgiou (1995) (see also the references therein).

On the other hand, the energy-to-peak gain filtering problem has received less attention. This kind of performance criterion has been discussed in Wilson (1989), where it is shown that the energy-to-peak gain can be

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computed from the controllability Grammian and the state-space representation of the system. In control system design, the problem of finding a controller such that the closed-loop gain from $\mathscr{L}_2$ to $\mathscr{L}_\infty$ (ℓ_2 to ℓ_∞) is below a prespecified level is called the generalized $\mathscr{H}_2$ control problem, since this optimization criterion reduces to the usual $\mathscr{H}_2$ norm when the controlled output is a scalar (see Rotea (1993) and also Scherer (1995) for details). The objective of the $\mathscr{L}_2$–$\mathscr{L}_\infty$ (ℓ_2–ℓ_∞) filter design problem is to minimize the peak value of the estimation error for all possible bounded energy disturbances. In this sense, the energy-to-peak filtering can be viewed as a deterministic formulation of the Kalman filter (see Grigoriadis & Watson, 1997). This strategy has been used for both full and reduced order filter design through $\mathscr{LMI}$s (Grigoriadis & Watson, 1997; Watson & Grigoriadis, 1997) and also as a criterion for model reduction (Grigoriadis, 1997). However, to the authors' knowledge, only precisely known systems have been addressed.

In this paper, an $\mathscr{LMI}$ solution to the guaranteed energy-to-peak filtering problem is proposed for both continuous-time and discrete-time systems. Using as starting point the state-space energy-to-peak gain computation from Wilson (1989) and standard output feedback control results of Scherer, Gahinet & Chilali (1997), necessary (in the sense that the filtering error system is quadratically stable) and sufficient conditions for the guaranteed $\mathscr{L}_2$–$\mathscr{L}_\infty$ full order filtering design are provided in terms of $\mathscr{LMI}$s for continuous-time systems with uncertainties in convex bounded domains. It is important to stress that several algebraic manipulations and appropriate change of variables are needed in order to obtain a convex formulation for the robust filtering problem. A similar strategy is used to provide equivalent results for discrete-time systems.

The contributions can be summarized as follows. The paper presents an $\mathscr{LMI}$ formulation for linear filter design, which can be immediately extended to handle the problem of robust filtering for linear systems with polytope-type uncertainties. In this context, the energy-to-peak gain is introduced as optimization criterion, allowing the guaranteed cost robust filter design to be performed through convex programming entirely based on $\mathscr{LMI}$s. Continuous-time as well as discrete-time uncertain linear systems with polytopic uncertainty are addressed; in both cases, the problems to be solved are convex optimization problems.

The paper is organized as follows: in the next section the robust filtering problem with guaranteed energy-to-peak performance is stated. In Section 3, the robust $\mathscr{L}_2$–$\mathscr{L}_\infty$ (continuous-time) and ℓ_2–ℓ_∞ (discrete-time) guaranteed filtering designs are addressed; the existence and parametrization of a robust filter is established in terms of necessary and sufficient $\mathscr{LMI}$s conditions. Numerical examples and final remarks conclude the paper.

1.1. Notation

The notation used in this paper is as follows: $\delta(t)$ indicates $\dot{x}(t)$ for continuous-time systems and $x(t+1)$ for discrete-time systems; L_2 denotes both $\mathscr{L}_2$ (continuous-time) and ℓ_2 (discrete-time), as well as L_∞ is used for both $\mathscr{L}_\infty$ (continuous-time) and ℓ_∞ (discrete-time) spaces. The boldface characters $\mathbf{I}$ and $\mathbf{0}$ denote, respectively, the identity and the null matrices of convenient sizes.

As discussed in Wilson (1989), $\mathscr{L}_p^n[0,\infty)$ denotes the Lebesgue space of measurable functions f from $[0,\infty)$ to $\mathbb{R}^n$ which satisfy

$$\|f\|_{\mathscr{L}_{p,r}} \triangleq \begin{cases} (\int_0^\infty \|f(t)\|_r^p \,\mathrm{d}t)^{1/p} < \infty & \text{for } 1 \le p < \infty, \\ \sup_t \|f(t)\|_r < \infty & \text{for } p = \infty, \end{cases} \tag{1}$$

where the usual vector r-norm on $\mathbb{R}^n$, i.e., $\|\cdot\|_r$ is defined as

$$\|f\|_r \triangleq \begin{cases} \left(\sum_{i=0}^{n} |f_i(t)|^r\right)^{1/r} & \text{for } 1 \le r < \infty, \\ \max_{i\in[1,n]} |f_i(t)| & \text{for } r = \infty. \end{cases} \tag{2}$$

Now, considering a discrete-time setting, $\mathscr{X}^n$ denotes the space of $\mathbb{R}^n$-valued sequences defined on the time set $\{0,1,2,\ldots\}$; ℓ_p denotes the set of all sequences ξ in $\mathscr{X}^n$ which satisfy

$$\|\xi\|_{\ell_{p,r}} \triangleq \begin{cases} \left(\sum_{t=0}^{\infty} \|\xi(t)\|_r^p\right)^{1/p} < \infty & \text{for } 1 \le p < \infty, \\ \sup_t \|\xi(t)\|_r < \infty & \text{for } p = \infty. \end{cases} \tag{3}$$

In Wilson (1989), explicit formulas for the induced norm of the bounded linear operator $G: \mathscr{L}_{2,r}^{n_1} \to \mathscr{L}_{\infty,r}^{n_2}$ for $r = 2$ and ∞ can be found. In this paper, only the Euclidean norm ($r = 2$) is considered; to avoid cumbersome notation, $\mathscr{L}_p$ (ℓ_p) is used instead of $\mathscr{L}_{p,2}$ ($\ell_{p,2}$).

2. Preliminaries

Consider the linear time-invariant system given by

$$(\mathscr{S}) \begin{cases} \delta x(t) = Ax(t) + Bw(t), \quad x(0) = x_o \\ y(t) = Cx(t) + Dw(t), \\ z(t) = Lx(t), \end{cases}$$

where $x(t): \mathbb{R} \to \mathbb{R}^n$ is the state vector, $y(t): \mathbb{R} \to \mathbb{R}^r$ is the measurement output vector, $w(t): \mathbb{R} \to \mathbb{R}^m$ is the noise signal vector (including process and measurement noises) and $z(t): \mathbb{R} \to \mathbb{R}^p$ is the signal to be estimated. The initial state condition x_o is considered to be known and, without loss of generality, assumed to be zero (see Khargonekar et al., 1996).

The system matrices are assumed to be unknown (uncertain) but belonging to a known convex compact set of polytope type, i.e.,

$(A, B, C, D, L) \in \mathscr{D}$,

where

$$\mathscr{D} \triangleq \Bigg\{(A, B, C, D, L) | (A, B, C, D, L) = \sum_{i=1}^{\kappa} \tau_i (A_i, B_i, C_i, D_i, L_i);\ \tau_i \geq 0; \sum_{i=1}^{\kappa} \tau_i = 1\Bigg\} \quad (4)$$

and $\mathscr{V}(\cdot)$ denotes the set of i, $i = 1, \ldots, \kappa$ vertices of the above convex polytope. This kind of convex bounded parameter uncertainty has been fairly investigated (see, for instance, Peres, Geromel & Bernussou (1993), Palhares, Takahashi & Peres (1997) and references therein).

The following assumption is made:

(A-i) The system $(\mathscr{S})$ is quadratically stable.

This assumption guarantees the boundedness of the estimation error, since the asymptotic stability of the error dynamics depends also on the states of the system $(\mathscr{S})$.

The purpose of the robust filtering problem is to find an estimate $\hat{z}(t)$ of the signal $z(t)$ such that a guaranteed performance criterion is minimized in an estimation error sense. For that, the aim is to design an *admissible* filter, i.e., an asymptotically stable linear filter, described by

$$(\mathscr{F}) \begin{cases} \delta\hat{x}(t) = A_f \hat{x}(t) + B_f y(t), & \hat{x}(0) = 0, \\ \hat{z}(t) = C_f \hat{x}(t), \end{cases}$$

where $\hat{x}(t): \mathbb{R} \to \mathbb{R}^{n_f}$. Moreover, the order of the filter is assumed to be equal to the order of the system ($n_f = n$).

A new augmented state vector can be defined

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

such that the filtering error dynamics is given by

$$(\mathscr{S}_{\mathscr{F}}) \begin{cases} \delta\tilde{x}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}w(t), & \tilde{x}(0) = \mathbf{0} \\ \tilde{z}(t) = \tilde{C}\tilde{x}(t), \end{cases}$$

where the filtering error output signal is denoted by $\tilde{z}(t) \triangleq z(t) - \hat{z}(t)$, with

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ B_f C & A_f \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ B_f D \end{bmatrix}, \quad \tilde{C} = [L \quad -C_f]. \quad (5)$$

The guaranteed cost filtering design problem to be investigated in this paper is:

(P_2^{∞}) *The robust L_2–L_∞ filtering problem: determine a stable linear filter $(\mathscr{F})$ ensuring a prespecified energy-to-peak gain, i.e.,*

$$\sup_{0 \neq w \in L_2} \frac{\|\tilde{z}\|_{L_\infty}}{\|w\|_{L_2}} < \gamma, \quad \forall (A, B, C, D, L) \in \mathscr{D} \quad (6)$$

such that the filtering error system $(\mathscr{S}_{\mathscr{F}})$ remains quadratically stable with the L_2–L_∞ gain limited by γ.

If the minimum value of γ such that problem (P_2^{∞}) is solvable is achieved, the optimal (under the assumption that system $(\mathscr{S})$ is quadratically stable) guaranteed L_2–L_∞ filter is obtained.

3. L_2–L_∞ guaranteed filtering design

The continuos- and discrete-time state-space characterizations of the energy-to-peak gain presented next can be viewed as the starting point for the main results of this section. First, consider $(A, B, C, D, L) \in \mathscr{D}$ arbitrary but fixed.

Lemma 3.1. *Let $\gamma > 0$ be given and assume that the filtering error system is stable. The L_2–L_∞ gain of $(\mathscr{S}_{\mathscr{F}})$ is limited by γ, i.e.,*

$$\sup_{0 \neq w \in L_2} \frac{\|\tilde{z}\|_{L_\infty}}{\|w\|_{L_2}} < \gamma \quad (7)$$

if and only if there exists $P = P' > \mathbf{0}$, $P \in \mathbb{R}^{2n \times 2n}$ such that

$$\tilde{C}P\tilde{C}' < \gamma^2 \mathbf{I}, \quad (8)$$

$$\Theta(P) < \mathbf{0}, \quad (9)$$

where $\Theta(P) \triangleq \tilde{A}P + P\tilde{A}' + \tilde{B}\tilde{B}'$ for continuous-time systems or $\Theta(P) \triangleq \tilde{A}P\tilde{A}' - P + \tilde{B}\tilde{B}'$ for discrete-time systems.

Proof. See Wilson (1989), Grigoriadis and Watson (1997), Skelton, Iwasaki and Grigoriadis (1997). □

Definition 3.1. The filtering error system $(\mathscr{S}_{\mathscr{F}})$ is said to be quadratically stable[1] with the L_2–L_∞ gain limited by γ if and only if there exists $P = P' > \mathbf{0}$, $P \in \mathbb{R}^{2n \times 2n}$ satisfying (8) and (9) for all $(\tilde{A}, \tilde{B}, \tilde{C}) \in \mathscr{D}$.

The following theorem plays an important role in the robust L_2–L_∞ filtering design:

Theorem 3.1. *Consider assumption* (A-i) *and let $\rho \triangleq \gamma^2 > 0$ be given*; $(\mathscr{F})$ *is an admissible filter assuring* (7) *if and only*

[1] This definition is a direct extension of the well-known concept of quadratic stability with a prescribed $\gamma > 0$ disturbance attenuation in the robust $\mathscr{H}_\infty$ control theory (Xie, 1996; Peres et al, 1993).

if there exist $R = R' > \mathbf{0}$, $R \in \mathbb{R}^{n\times n}$, $X = X' > \mathbf{0}$, $X \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{p\times n}$ *and* $Z \in \mathbb{R}^{n\times r}$ *satisfying*

$$\Lambda(L, R, X, N, \rho) > \mathbf{0}, \tag{10}$$

$$\Gamma(A, B, C, D, R, X, M, Z) > \mathbf{0}, \tag{11}$$

where

$$\Lambda(L, R, X, N, \rho) \triangleq \begin{bmatrix} \rho\mathbf{I} & L & L-N \\ L' & R & X \\ L'-N' & X & X \end{bmatrix} \tag{12}$$

and

$$\Gamma(A, B, C, D, R, X, M, Z) \triangleq \begin{bmatrix} -A'R-RA-ZC-C'Z' & -A'X-RA-ZC-M & -RB-ZD \\ -A'R-XA-C'Z'-M' & -A'X-XA & -XB \\ -B'R-D'Z' & -B'X & \mathbf{I} \end{bmatrix} \tag{13}$$

for continuous-time systems or

$$\Gamma(A, B, C, D, R, X, M, Z) \triangleq \begin{bmatrix} R & X & RA+ZC & RA+ZC+M & RB+ZD \\ X & X & XA & XA & XB \\ A'R+C'Z' & A'X & R & X & \mathbf{0} \\ A'R+C'Z'+M' & A'X & X & X & \mathbf{0} \\ B'R+D'Z' & B'X & \mathbf{0} & \mathbf{0} & I \end{bmatrix} \tag{14}$$

for discrete-time systems.

Proof (*Continuous-time case*). (*Necessity*) Considering assumption (A-i), let $\rho \triangleq \gamma^2 > 0$ be given and assume there exists a stable linear filter ($\mathscr{F}$) such that (7) is guaranteed. Thus, from Lemma 3.1, there exists $P = P' > \mathbf{0}$ satisfying (8) and (9). Let P and its inverse, denoted by S, be partitioned as

$$P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P'_{12} & P_{22} \end{bmatrix}, \quad S \triangleq \begin{bmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{bmatrix} > \mathbf{0}. \tag{15}$$

From the identity $PS = \mathbf{I}$ it follows that

$$\begin{bmatrix} P_{11}S_{11} + P_{12}S'_{12} & P_{11}S_{12} + P_{12}S_{22} \\ P'_{12}S_{11} + P_{22}S'_{12} & P'_{12}S_{12} + P_{22}S_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{16}$$

with

$$P_{11} > \mathbf{0}, \quad S_{11} > \mathbf{0} \tag{17}$$

and the identity $\mathbf{I} - S_{11}P_{11} = S_{12}P'_{12}$ (or equivalently $\mathbf{I} - P_{11}S_{11} = P_{12}S'_{12}$) can be constructed such that S_{12} and P_{12} are square and nonsingular matrices (see Scherer et al., 1997; Chilali & Gahinet, 1996) and thus $\mathbf{I} - S_{11}P_{11}$ is invertible. Therefore, define the nonsingular matrices

$$J \triangleq \begin{bmatrix} S_{11} & \mathbf{I} \\ S'_{12} & \mathbf{0} \end{bmatrix}, \quad \tilde{J} \triangleq \begin{bmatrix} \mathbf{I} & P_{11} \\ \mathbf{0} & P'_{12} \end{bmatrix}, \tag{18}$$

where $PJ = \tilde{J}$ holds and also

$$\hat{J}_1 = \mathrm{diag}[J, \mathbf{I}] \in \mathbb{R}^{(2n+m)\times(2n+m)}, \quad \hat{J}_2 = \mathrm{diag}[\mathbf{I}, J] \in \mathbb{R}^{(p+2n)\times(p+2n)}. \tag{19}$$

Now, from Schur's complement (see Albert, 1969), it follows that the $\mathscr{LMI}s$ (8) and (9) are, respectively, equivalent to

$$\begin{bmatrix} \rho\mathbf{I} & \tilde{C}P \\ P\tilde{C}' & P \end{bmatrix} > \mathbf{0} \tag{20}$$

and

$$\begin{bmatrix} \tilde{A}P + P\tilde{A}' & \tilde{B} \\ \tilde{B}' & -\mathbf{I} \end{bmatrix} < \mathbf{0}. \tag{21}$$

Let the $\mathscr{LMI}$ (21) be pre- and postmultiplied by $\hat{J}'_1$ and $\hat{J}_1$, respectively, such that

$$\begin{bmatrix} A'S_{11} + S_{11}A + S_{12}B_fC + C'B'_fS'_{12} & A' + S_{11}AP_{11} + S_{12}B_fCP_{11} + S_{12}A_fP'_{12} & S_{11}B + S_{12}B_fD \\ A + P_{11}A'S_{11} + P_{11}C'B'_fS'_{12} + P_{12}A'_fS'_{12} & \mathrm{AP}_{11} + P_{11}A' & B \\ B'S_{11} + D'B'_fS'_{12} & \mathrm{B}' & -\mathbf{I} \end{bmatrix} < \mathbf{0}. \tag{22}$$

Define the change of variable,

$$Z \triangleq S_{12}B_f \quad \text{and} \quad \hat{Z} \triangleq S_{12}A_fP'_{12} \tag{23}$$

and construct the transformation

$$Q_1 \triangleq \mathrm{diag}[\mathbf{I}, P_{11}^{-1}, \mathbf{I}]. \tag{24}$$

Now, pre- and postmultiplying (22) by Q_1, with (23), one gets

$$\begin{bmatrix} A'S_{11} + S_{11}A + ZC + C'Z' & A'P_{11}^{-1} + S_{11}A + ZC + \hat{Z}P_{11}^{-1} & S_{11}B + ZD \\ P_{11}^{-1}A + A'S_{11} + C'Z' + P_{11}^{-1}\hat{Z}' & A'P_{11}^{-1} + P_{11}^{-1}A & P_{11}^{-1}B \\ B'S_{11} + D'Z' & B'P_{11}^{-1} & -\mathbf{I} \end{bmatrix} < \mathbf{0} \tag{25}$$

and a new change of variables defined by

$$R \triangleq S_{11}, \quad X \triangleq P_{11}^{-1}, \quad M \triangleq \hat{Z}X \tag{26}$$

yields the $\mathscr{LMI}$ (11).

Similar manipulations can be performed in the $\mathscr{LMI}$ (20): pre- and postmultiplying (20) by $\hat{J}_2'$ and $\hat{J}_2$, respectively, taking into account the change of variable

$$\tilde{Z} \triangleq C_f P_{12}' \tag{27}$$

and pre- and postmultiplying by the transformation

$$Q_2 \triangleq \mathrm{diag}[\mathbf{I}, \mathbf{I}, P_{11}^{-1}] \tag{28}$$

one gets

$$\begin{bmatrix} \rho\mathbf{I} & L & L - \tilde{Z}P_{11}^{-1} \\ L' & S_{11} & P_{11}^{-1} \\ L' - P_{11}^{-1}\tilde{Z}' & P_{11}^{-1} & P_{11}^{-1} \end{bmatrix} > \mathbf{0}. \tag{29}$$

Using (26) and defining another change of variable given by

$$N \triangleq \tilde{Z}X \tag{30}$$

the $\mathscr{LMI}$ (10) follows immediatly.

(Sufficiency). Assume there exist feasible matrices R, X, M, N and Z satisfying (10) and (11). Then, the sufficient part of the proof follows in a straightforward way from the change of variables (30) and (26) with $R - X > \mathbf{0}$, implying that $\mathbf{I} - S_{11}P_{11}$ is nonsingular[2] and the identity $\mathbf{I} - S_{11}P_{11} = S_{12}P_{12}'$ guarantees that the matrices P_{12}' and S_{12} are invertible such that J and $\tilde{J}$ defined in (18) are nonsingular. Taking into account matrices P_{12}' and S_{12}, (23) and (27) can be uniquely solved for the filter matrices A_f, B_f, C_f and thus the equivalences between (25) and

$$\begin{bmatrix} J'\tilde{A}\tilde{J} + \tilde{J}'\tilde{A}'J & J'\tilde{B} \\ \tilde{B}'J & -\mathbf{I} \end{bmatrix} < \mathbf{0} \tag{31}$$

and between (29) and

$$\begin{bmatrix} \rho\mathbf{I} & \tilde{C}\tilde{J} \\ \tilde{J}'\tilde{C} & \tilde{J}'J \end{bmatrix} > \mathbf{0} \tag{32}$$

are established.

Since J is invertible, inequalities (8) and (9) can be obtained by Schur's complement after pre- and postmultiplying (31) and (32), respectively, by $\mathrm{diag}[J^{-1}, \mathbf{I}]$ and $\mathrm{diag}[\mathbf{I}, J^{-1}]$. Hence, the filter constructed guarantees the bound $\|\tilde{z}\|_{\mathscr{L}_\infty} < \sqrt{\rho}\|w\|_{\mathscr{L}_2}$.

(*Discrete-time Case*). (Necessity) The $\mathscr{LMI}$ (11) is established applying the Schur complement to (9), yielding

$$\begin{bmatrix} P & \tilde{A}P & \tilde{B} \\ P\tilde{A}' & P & \mathbf{0} \\ \tilde{B}' & \mathbf{0} & \mathbf{I} \end{bmatrix} > \mathbf{0}. \tag{33}$$

Further, defining

$$\hat{J} = \mathrm{diag}[J, J, \mathbf{I}] \in \mathbb{R}^{(4n+m)\times(4n+m)} \tag{34}$$

with J defined in (18), pre- and postmultiplying (33) by $\hat{J}'$ and $\hat{J}$, respectively, and taking into account the change of variables (23), it follows that

$$\begin{bmatrix} S_{11} & \mathbf{I} & S_{11}A + ZC & \cdots & S_{11}AP_{11} + ZCP_{11} + \hat{Z} & S_{11}B + ZD \\ \mathbf{I} & P_{11} & A & \cdots & AP_{11} & B \\ A'S_{11} + C'Z' & A' & S_{11} & \cdots & \mathbf{I} & \mathbf{0} \\ P_{11}A'S_{11} + P_{11}C'Z' + \hat{Z}' & P_{11}A' & \mathbf{I} & \cdots & P_{11} & \mathbf{0} \\ B'S_{11} + D'Z' & B' & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} \end{bmatrix} > \mathbf{0}. \tag{35}$$

Now, pre- and postmultiplying the $\mathscr{LMI}$ (35) by

$$\mathrm{diag}[\mathbf{I}, P_{11}^{-1}, \mathbf{I}, P_{11}^{-1}, \mathbf{I}]$$

and considering the change of variables (26), inequality (11) is obtained.

(*Sufficiency*). Follows in a straightforward way. □

Remark 3.1. It must be emphasized that $\Gamma(\cdot)$ and $\Lambda(\cdot)$ are affine with respect to all the matrices involved, allowing the immediate extension of the results to deal with uncertain systems in convex bounded domains, requiring the investigation of the $\mathscr{LMI}$ at the vertices of the uncertainty polytope only. Furthermore, standard $\mathscr{LMI}$ programming tools can be used to test the existence of admissible filters.

From the above results, the next theorem provides a parametrization of admissible L_2–L_∞ filters.

Theorem 3.2. *Any matrices R, X, M, N and Z satisfying Theorem* 3.1 *yield admissible L_2–L_∞ filters ($\mathscr{F}$) given by*

$$\begin{aligned} A_f &= (X - R)^{-1}M, \\ B_f &= (X - R)^{-1}Z, \\ C_f &= N. \end{aligned} \tag{36}$$

[2] Note that the constraint $\left[\begin{smallmatrix} R & X \\ X & X \end{smallmatrix}\right] > \mathbf{0}$ or, equivalently, $R - X = S_{11} - P_{11}^{-1} > \mathbf{0}$ is embedded in the $\mathscr{LMI}$ (10).

Proof. From (23) and (27), it follows that $A_f = S_{12}^{-1}\hat{Z}(P'_{12})^{-1}$, $C_f = \tilde{Z}(P'_{12})^{-1}$, $B_f \triangleq S_{12}^{-1}Z$ and the transfer matrix of the measurement output $y(t)$ to the estimation output $\hat{z}(t)$ of the filter $(\mathscr{F})$, is given by

$$H_{\hat{z}y}(\zeta) = \tilde{Z}(\zeta(S_{12}P'_{12}) - \hat{Z})^{-1}Z, \tag{37}$$

where ζ denotes both the continuous- and discrete-time frequency domain variable. Applying (16), (37) yields

$$H_{\hat{z}y}(\zeta) = \tilde{Z}(\zeta(\mathbf{I} - S_{11}P_{11}) - \hat{Z})^{-1}Z. \tag{38}$$

Finally, with the change of variables defined in (26) and (30), the transfer matrix of the filter $(\mathscr{F})$

$$H_{\hat{z}y}(\zeta) = N(\zeta(X - R) - M)^{-1}Z \tag{39}$$

is obtained. □

In order to extend the results of Theorem 3.1 to the uncertain case, define the following sets:

$$\mathscr{T} \triangleq \{(R, X, M, N, Z, \rho) \text{ satisfying Theorem 3.1 } \forall (A, B, C, D, L) \in \mathscr{D}\}, \tag{40}$$

$$\mathscr{T}_v \triangleq \{(R, X, M, N, Z, \rho) \text{ satisfying Theorem 3.1 } \forall (A_i, B_i, C_i, D_i, L_i) \in \mathscr{V}(\mathscr{D})\}. \tag{41}$$

From the affinity of the above $\mathscr{LMI}$s, it is clear that the equivalence $\mathscr{T} \equiv \mathscr{T}_v$ holds, implying that it suffices to verify the constraints only at the vertices of the polytope of uncertain parameters. The next theorem characterizes the robust L_2–L_∞ filter design for a prescribed positive level γ.

Theorem 3.3. *Consider assumption* (A-i) *and let* $\rho = \gamma^2 > 0$ *be given. The problem* (P_2^∞) *is solvable if and only if there exist* $R = R' > \mathbf{0}$, $R \in \mathbb{R}^{n\times n}$, $X = X' > \mathbf{0}$, $X \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{p\times n}$ *and* $Z \in \mathbb{R}^{n\times r}$ *such that*

$$(R, X, M, N, Z, \rho) \in \mathscr{T}_v. \tag{42}$$

In the affirmative case, the admissible filter $(\mathscr{F})$ *is given by* (36).

Proof. The proof follows in a straightforward way from the proof of Theorem 3.1 and Definition 3.1 together with the equivalence $\mathscr{T} \equiv \mathscr{T}_v$. □

Therefore, the optimal L_2–L_∞ guaranteed filtering cost is stated in the following corollary by means of an $\mathscr{LMI}$ optimization procedure.

Corollary 3.1. *The solution of*

$$\min_{R,X,M,N,Z,\rho} \ \rho \quad \text{s.t.} \quad (R, X, M, N, Z, \rho) \in \mathscr{T}_v \tag{43}$$

is such that $\sqrt{\rho}$ *is the minimum* L_2–L_∞ *guaranteed filtering cost, i.e.,*

$$\sup_{0 \neq w \in L_2} \frac{\|\tilde{z}\|_{L_\infty}}{\|w\|_{L_2}} < \sqrt{\rho}, \quad \forall (A, B, C, D, L) \in \mathscr{D}.$$

The matrices of the admissible filter are given by (36).

Proof. The proof follows in a straightforward way. □

4. Examples

Example 1. Consider the following second-order resonant system (borrowed from Grigoriadis and Watson (1997)):

$$\dot{x}(t) = \begin{bmatrix} 0 & 11 \\ -11 & -2.2 + \alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 + \beta & 0 \end{bmatrix} w(t),$$
$$y(t) = [0 \quad 1]x(t) + [0 \quad 1 + \beta]w(t), \tag{44}$$
$$z(t) = [1 \quad 0]x(t).$$

As pointed out in Grigoriadis and Watson (1997), for $\alpha = 0$ and $\beta = 0$, this system model corresponds to a vibrating system with natural frequency $\omega_n = 11$ rad/s and damping ratio $\zeta = 0.1$. The position measurement $y(t)$ is corrupted by noise and the objective is to estimate a velocity signal $z(t)$.

Assuming that the input disturbance is of bounded energy type and that the aim is to minimize the output peak value, the optimal $\mathscr{L}_2$–$\mathscr{L}_\infty$ filtering level is given by $\gamma^* = 0.4654$ with the filter matrices

$$A_f = \begin{bmatrix} -0.4506 & 10.9928 \\ -10.9267 & -1.9662 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.0056 \\ 0.2715 \end{bmatrix},$$
$$C_f = [0.7956 \quad -0.0165]. \tag{45}$$

Considering other scenery, i.e., white noise input disturbances, the standard $\mathscr{H}_2$ filtering design (see, for example, Shaked and de Souza (1994)) furnishes as optimal cost $J^* = 0.4654$. As pointed out in Bernstein (1992) the optimal $\mathscr{H}_2$ filtering cost is an upper bound to the optimal $\mathscr{L}_2$–$\mathscr{L}_\infty$ filtering cost: $\gamma^* \leq J^*$.

Now consider (44) as an uncertain system with $-1 \leq \alpha \leq 1$ (meaning that $\zeta \in [-0.1455; -0.0545]$) and $0 \leq \beta \leq 1$ (i.e., the amplitude of the bounded energy input can vary), yielding an uncertain systems of $\kappa = 4$ vertices. The optimal $\mathscr{L}_2$–$\mathscr{L}_\infty$ guaranteed filtering cost obtained is $\gamma^* = 1.2034$, with the robust filter described by

$$A_f = \begin{bmatrix} -0.2673 & 10.9894 \\ -10.9791 & -1.2951 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.0078 \\ 0.6434 \end{bmatrix},$$
$$C_f = [0.5625 \quad -0.0068]. \tag{46}$$

Table 1
Computed $\mathscr{L}_2$–$\mathscr{L}_\infty$ filtering costs for the 4 vertices and $\alpha=\beta=0$ (nominal system) with the robust filter (46) connected to the uncertain system (44)

	$\alpha=-1$ $\beta=0$	$\alpha=-1$ $\beta=1$	$\alpha=1$ $\beta=0$	$\alpha=1$ $\beta=1$	$\alpha=\beta=0$
γ	0.4209	0.8419	0.6017	1.2034	0.4802

Table 2
Computed ℓ_2–ℓ_∞ filtering costs at vertices κ, $\kappa=1,\dots,8$ with the robust filter (48) and (49) connected to the uncertain system (47)

	$\eta=0.90$	$\eta=0.90$	$\eta=0.90$	$\eta=0.90$
	$\tau=0.0079$	$\tau=0.0079$	$\tau=0.0794$	$\tau=0.0794$
	$\phi=0$	$\phi=1$	$\phi=0$	$\phi=1$
γ	0.8080	0.8056	1.0993	0.8863
	$\eta=0.99$	$\eta=0.99$	$\eta=0.99$	$\eta=0.99$
	$\tau=0.0079$	$\tau=0.0079$	$\tau=0.0794$	$\tau=0.0794$
	$\phi=0$	$\phi=1$	$\phi=0$	$\phi=1$
γ	1.0096	0.9429	6.1795	5.0007

Table 1 shows the calculated $\mathscr{L}_2$–$\mathscr{L}_\infty$ filtering costs for the four vertices and also $\alpha=\beta=0$ (nominal system) considering the robust filter (46) connected to the uncertain system (44) (the effectiveness of the computed γ^* guaranteed bound is apparent).

Example 2. Consider the following discrete-time linear system (also investigated in Rauch, Tung and Striebel (1965)), with $w(t)\in\ell_2$:

$$x(t+1)=\begin{bmatrix}\eta & 1 & 0.5 & 0.5\\ 0 & \eta & 1 & 1\\ 0 & 0 & \eta & 0\\ 0 & 0 & 0 & 0.606\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\0\\\tau\end{bmatrix}w(t),$$

$$y(t)=[1 \quad \phi \quad 0 \quad 0]x(t)+w(t), z(t)=[1 \quad 1 \quad 0 \quad 0]x(t). \tag{47}$$

Assume that some parameters are unknown but bounded, i.e.,

$$0.90\le\eta\le 0.99,\quad 0.0079\le\tau\le 0.0794,\quad 0\le\phi\le 1.$$

The optimal ℓ_2–ℓ_∞ guaranteed filtering cost is given by $\gamma^*=23.9745$ with the filter matrices

$$A_f=\begin{bmatrix}0.1181 & 1.2908 & -60.1624 & 0.9150\\ -0.0403 & 0.8707 & 1.1952 & 1.6683\\ 0 & -0.0009 & 0.9644 & 0.0128\\ -0.0014 & 0.0063 & 1.3000 & 0.6042\end{bmatrix},$$

$$B_f=\begin{bmatrix}0.8605\\0.0416\\0\\0.0014\end{bmatrix}, \tag{48}$$

$$C_f=[0.9687 \quad -0.8277 \quad -12.2220 \quad -1.2444]. \tag{49}$$

Table 2 presents the calculated ℓ_2–ℓ_∞ filtering costs for the eight vertices with the robust filter (48) and (49) connected to the uncertain system (47); once again, γ^* characterizes a guaranteed filtering cost.

5. Conclusions

The robust filtering problem with guaranteed energy-to-peak performance for continuous- and discrete-time uncertain linear systems in convex bounded domains has been addressed through an $\mathscr{LMI}$ approach. Necessary and sufficient conditions are obtained for full order filtering, allowing the problem to be solved through convex optimization procedures (global convergence, efficient algorithms).

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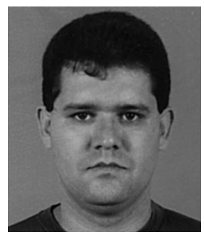

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