A GENERAL QUADRATURE FORMULA FOR EFFICIENT EVALUATION OF RADIATION INTEGRALS

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Abstract

An efficient and general numerical procedure for evaluating the Physical Optics integrals occurring in the analysis and design of reflector antennas is presented. The procedure combines Ludwig's integration technique (i.e., independent interpolation of the integrand amplitude and phase) with a predictor-corrector scheme. This combination eliminates the amplitude and phase ambiguities present in the real+imaginary algebra used in machine calculations. The performance of the procedure is investigated on a typical case study: the evaluation of the field scattered by an offset paraboloidal reflector.

A. Introduction

A common tool for the analysis and design of reflector antennas is the Physical Optics technique (PO) [1]. In this method an approximation for the reflector surface currents is used with the free-space dyadic Green's function to obtain a double integral that, once evaluated, yields the antenna radiation characteristics. The PO integrand is a complex function and, in most cases, has a slowly varying amplitude and a highly oscillatory phase. Due to this characteristic, the evaluation of these integrals using standard numerical techniques (e.g., Simpson's rule, Newton-Cotes formulas, Gaussian Quadratures, etc.), requires considerable computational effort. To overcome this difficulty, Ludwig proposed an algorithm [2] in which linear functions are used to approximate separately the amplitude and phase of the integrand over relatively small regions of the reflector surface (integration cells). The advantage of this method is that the integral can be evaluated in closed form over each integration cell, and hence the integrand oscillatory behavior no longer constitutes a strong factor in determining the size of these cells.

Although numerically efficient, Ludwig's algorithm is not generally applicable to the evaluation of radiation integrals. The reason for this resides on the fact that machine computations are performed using real and imaginary parts of complex numbers, as opposed to amplitude and phase. This feature introduces a $n2\pi$ radians ambiguity ($n = 0, \pm1, \pm2, ...$) in the phase values, which cannot be resolved if only the real and imaginary parts of the complex integrand are known. To eliminate this ambiguity, goemetrical optics concepts can be used to determine the integrand phase—it is determined by the path length from the feed system, to the reflector surface, to the observation point. However, this scheme is strongly dependent on the antenna geometry, and in many applications becomes difficult to apply (e.g., array feed). The final consequence is the loss of generality of Ludwig's algorithm.

This work presents a method to overcome the difficulties discussed in the previous paragraph. A predictor-corrector algorithm is introduced and incorporated in the Ludwig's integration scheme, allowing it to be used without specific integrand phase information. This produces a very general integration method, with all the advantages of Ludwig's algorithm and none of its shortcomings. In order to demonstrate the characteristics of the proposed algorithm, an offset paraboloidal reflector is considered and analyzed in this work. The radiated fields are evaluated to show that the proposed predictor-corrector algorithm is capable of extracting the correct value of the integrand phase, even in the presence of a highly oscillatory behavior.

B. The Predictor-Corrector Algorithm

The PO method yields integrals of the form

$$\iint_{S'} A(u,v)\, e^{-jkB(u,v)}\, du\, dv, \tag{1}$$

where S' is the illuminated area of the reflector surface, u and v are the corresponding integration variables, and $k = 2\pi/\lambda$. The above complex integrand has been represented in exponential form to explicitly show its amplitude and phase functions (both real) $A(u,v)$ and $kB(u,v)$, respectively. $A(u,v)$ and $B(u,v)$ depend on the incident field, the observation point location, and the geometry of the reflector surface. As mentioned previously, due to the fact that machine computations involving complex numbers are not performed in exponential form, the exact value of $kB(u,v)$ has a multiple of 2π radians ambiguity. Elimination of this ambiguity is performed using the predictor-corrector algorithm presented below.

The basic principle of the predictor-corrector is to estimate (predict) the amplitude and phase of the integrand at each cell node and then use these estimated values to eliminate (correct) any $n2\pi$ radians ($n = 0, \pm1, \pm2, ...$) ambiguity present. To accomplish this, the domain of integration is first divided into small cells, following the procedures of the Ludwig's algorithm [2]. The predictor step is done by using biquadratic extrapolation functions of the form:

$$A_c(u,v) = A_1u^2v^2 + A_2u^2v + A_3uv^2 + A_4u^2 + A_5v^2 + A_6uv + A_7u + A_8v + A_9, \tag{2}$$

A close observation of the above expansion shows that, to uniquely determine $A_c(u,v)$, nine independent values of $A(u,v)$ are required. For this reason the predictor-corrector cell is composed by four adjacent Ludwig's algorithm cells (see Fig. 1). The subscript c is used in the above expression to stress the fact that the extrapolation function is independently evaluated on each predictor-corrector cell. Eq. (2) is used to extrapolate the value of the amplitude $A_c(u,v)$ from a cell to its adjacent one. The determination of the nine coefficients A_i of Eq. (2) can be easily done using the scheme described in Ref. [3]. A similar procedure is used to extrapolate the phase $kB_c(u,v)$.

The first step of the integration algorithm is to obtain the correct values (i.e., without phase ambiguity) of the integrand over an initial cell. Due to this requirement, it is crucial to start the predictor-corrector scheme with a sufficiently small cell to guarantee that the $A_c(u,v)$ amplitudes are not zero (being arbitrarily assumed as positive) and $kB_c(u,v)$ changes by less than π radians from any node to its adjacent ones. Using geometrical optics it can be shown that the phase difference between any two adjacent nodes is always less than $4\pi\ell/\lambda$, where ℓ is the physical distance between the nodes. An initial cell with sides equal to $0.1\,\lambda$ constitutes, then, a sufficiently small cell to start the predictor-corrector process. Using this small initial cell the integrand values can be predicted and corrected in progressively enlarging adjacent cells, until any desired cell size is attained (see Fig. 2). After determining the predicted and corrected integrand values over the first actual cell of the grid, they are used to predict and correct the values over the second cell, and so on.

To apply the predictor-corrector algorithm it is assumed that the integrand is unambiguously known at all nodes of a specific cell (C_1). It is also assumed that the integrand values are known at all nodes of an adjacent cell (C_2), apart from its amplitude sign and a $n2\pi$ radians ambiguity. The determination of the correct amplitude and phase values over this adjacent cell is performed using the extrapolation functions [given by Eq. (2)] accordingly to the following steps:

1 – The amplitude is extrapolated, using the amplitude values of the cell C_1, to determine its sign at all the nodes of the adjacent cell C_2. This step accounts for the fact that amplitude values can be either positive or negative, permitting the corresponding phase to vary continuously. The correct magnitude of the integrand is already known; only its sign remains to be determined, and this is done at this step.

2 – The integrand phase is known within a $n2\pi$ radians ambiguity in any node of the cell C_2. If the sign of the corresponding amplitude is predicted to be negative in step 1, a phase increment of π radians is added to the phase value. Note that if the value of the amplitude is very small, the algorithm may fail to predict the correct sign in step 1, causing a wrong π phase-shift to be added to the phase value. This problem is overcome in step 4.

3 – The phase is extrapolated, using the phase values of the cell C_1, and compared with the value determined in step 2. By adding to the step 2 value the $n2\pi$ radians that yields the smallest absolute difference between it and the predicted value, the unambiguous (corrected) phase value is determined.

4 – The predicted and corrected phases are compared. If the magnitude of their difference is grater than, say, $\pi/2$, the sign of the amplitude has been incorrectly predicted in step 1. The sign of the amplitude is then reversed, a π radians increment is added to the phase to make it correct in step 2, and step 3 is repeated.

The above steps 1 to 4 are repeated for all nodes on the reflector surface, using the appropriate adjacent cells C_1 and C_2. Once the integrand values have been determined unambiguously at all nodes, Ludwig's integration algorithm is used to determine the value of the radiation integral.

C. Numerical Results

Since the present work is primarily concerned with evaluating the performance of the combination Ludwig's plus predictor-corrector algorithms, numerical simulations were performed on radiation integrals having a rectangular integration domain. The test case of an offset paraboloidal reflector with focal length $F = 40\,\lambda$ and offset distance $d_0 = 40\,\lambda$, illuminated by a feed located at its focus, was selected (see Fig. 3). The projection of the reflector rim on a plane perpendicular to its axis is a rectangle with sides of $50\,\lambda$ by $100\,\lambda$. The feed is a perfectly linearly polarized ($x-$direction) spherical wave source radiating an amplitude pattern described by $\cos^E(\theta_F)$, where θ_F is the angle measured from the feed boresight direction and $E = 8.9$ (this feed approximately produces -10 dB of illumination taper at the reflector E–plane rim [$\phi = 0^\circ$]). The feed is aimed at the reflector point with coordinates $x = d_0$ and $y = 0$.

Because of the predictor-corrector cell characteristic, the total number of nodes in a certain direction of the grid must be an odd number. In the $x-$direction this number is represented by N. In the $y-$direction, it is represented by $2N + 1$. So, the total number of nodes, which is equal to the number of integrand evaluations, is given by $N(2N + 1)$.

Fig. 4 shows the best result obtained by the Ludwig's algorithm with the minimum number of nodes, $N = 11$. In this figure the actual phase $kB_c(u, v)$ of the integrand was obtained using geometrical optics and the amplitude $A_c(u, v)$ was always supposed positive (no predictor-corrector algorithm was used). We can observe discrepancies in the cross polarization pattern (always bellow -60 dB). These are due to the fact that the integrand amplitude was supposed positive, causing discontinuities of π radians in the phase.

Fig. 5 shows the performance of the Ludwig's and the predictor-corrector algorithms combined. In this case the integrand values were evaluated as real and imaginary parts, and the task of determining the correct phase was left to the predictor-corrector algorithm. Note that, under this circumstances, the Ludwig's algorithm was capable of performing the integral with the same number of nodes as in the previous case, $N = 11$ (this was the best result). As the correct amplitude and phase values were used, we do not observe any discrepancy in the cross polarization pattern.

D. Conclusions

This work presents a quadrature formula for evaluating radiation integrals. The quadrature formula is based on a combination of Ludwig's integration algorithm with a predictor-corrector technique. This combination eliminates the difficulties associated with the determination of the integrand phase, present when Ludwig's algorithm is used alone. It yields a very general, accurate, and numerically efficient technique for evaluating reflector antenna radiation integrals.

As a final comment it should be mentioned that the predictor-corrector technique is independent of the specific numerical interpolation used to evaluate the integral. For example, instead of the linear interpolation used in Ludwig's algorithm, one may select to use a biquadratic expansion for the integrand amplitude and phase [3]. In this case the same expansion would be used in both the predictor-corrector and the algorithm that actually determines the integral value. Also, triangular cells can be used; they are in general more efficient to perform integrations over surface areas with curved projected rims.

References

[1] W.V.T. Rusch and P.D. Potter, *Analysis of Reflector Antennas*, New York, Academic Press, 1971.

[2] A. C. Ludwig, "Computation of Radiation Patterns Involving Numerical Double Integration", IEEE Trans. Antennas and Propagat., **AP-16**, No. 6, pp. 767-769, November 1968.

[3] Glenn D. Crabtree, "A Numerical Quadrature Technique for Physical Optics Scattering Analysis", IEEE Trans. Magnetics, **27**, No. 5, pp. 4291-4294, September 1991.

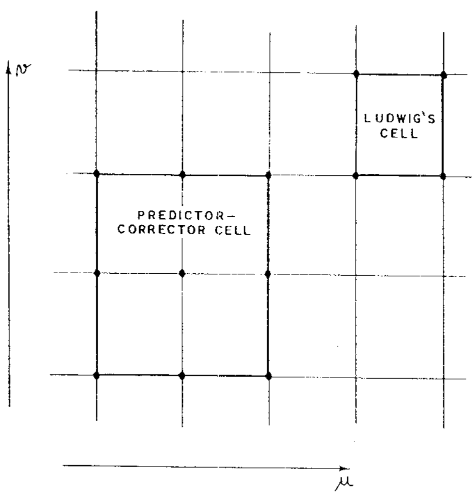

Fig. 1 – The geometry of the cells.

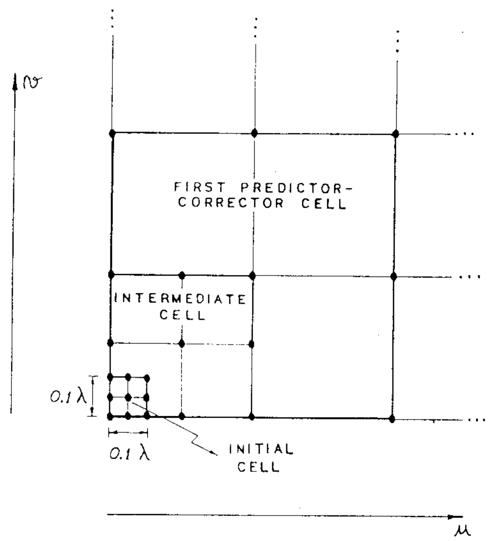

Fig. 2 – The initial predictor-corrector cells.

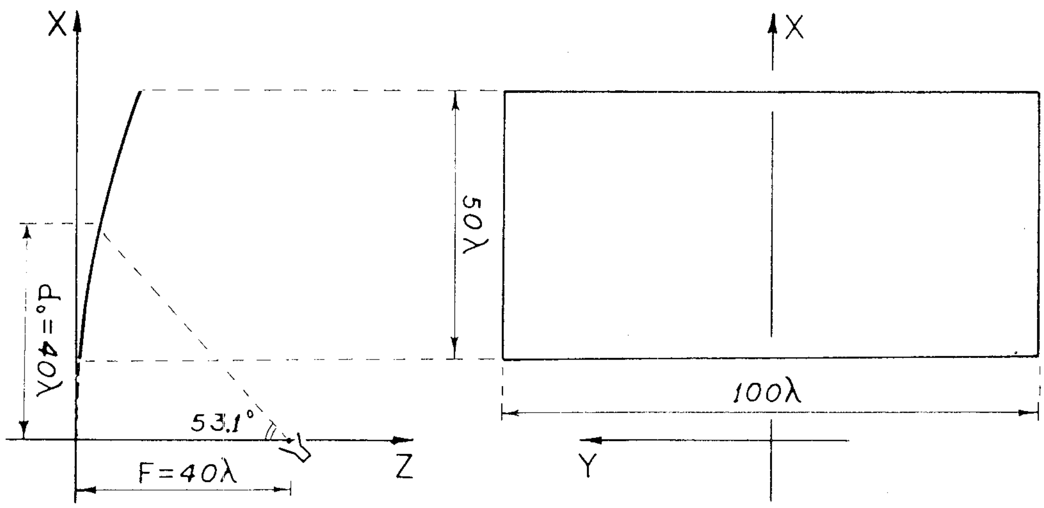

Fig. 3 – The geometry of the reflector.

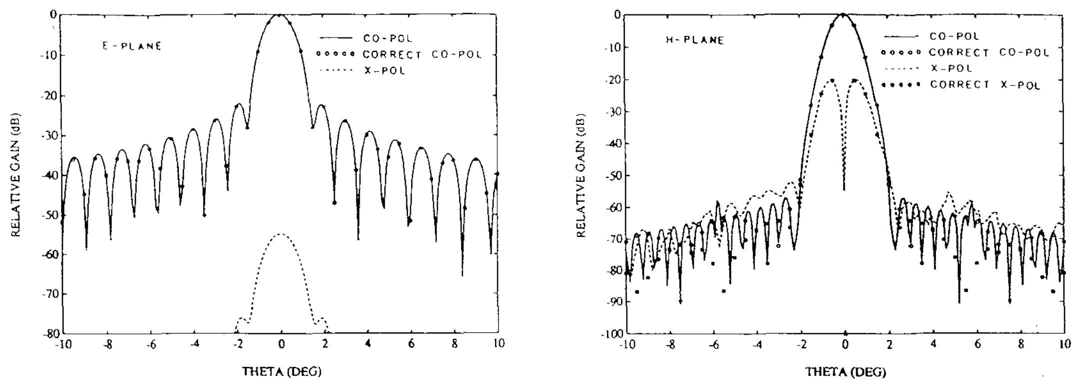

Fig. 4 – Reflector scattering: Ludwig's algorithm.

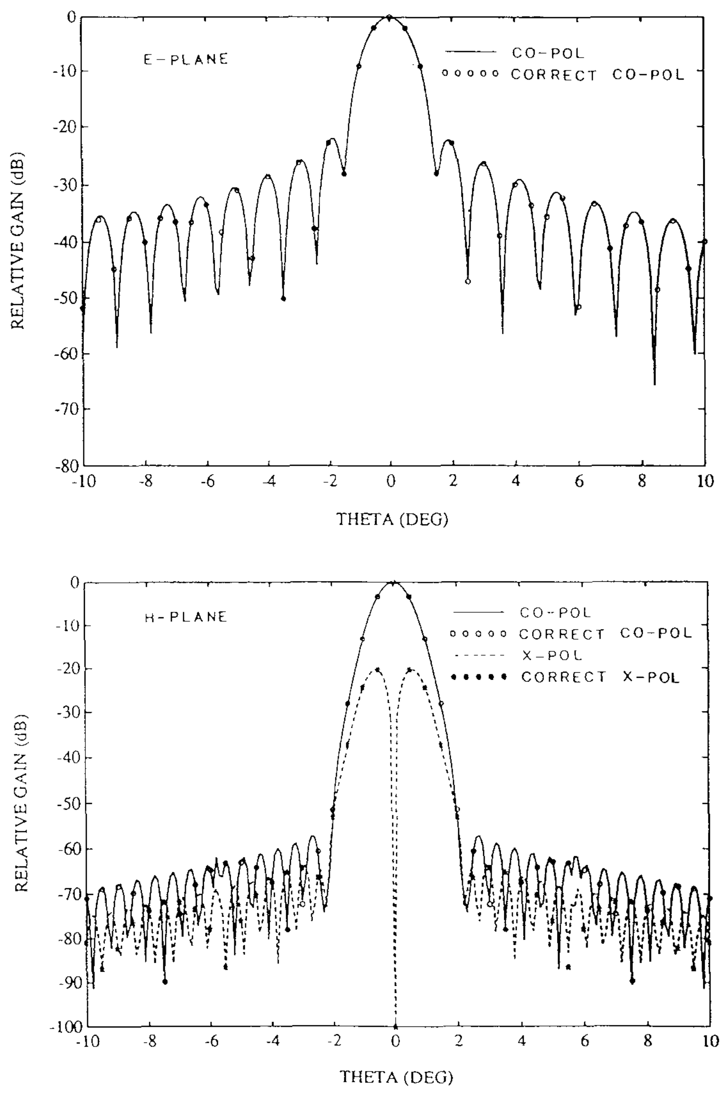

Fig. 5 – Reflector scattering: Ludwig's and the predictor-corrector algorithms.