The Inf-Sup Condition in Meshfree Analyses of Electromagnetism

Williams L. Nicomedes[1,2], Klaus-Jürgen Bathe[1], Renato C. Mesquita[3], and Fernando J. S. Moreira[2]

[1]Dept. Mechanical Engineering, Massachusetts Institute of Technology, MIT, 77 Mass. Avenue, Cambridge MA 02139, USA
[2]Dept. Electronic Engineering, Federal University of Minas Gerais, Av. Antonio Carlos 6627, Belo Horizonte MG, Brazil
[3]Dept. Electrical Engineering, Federal University of Minas Gerais, Av. Antonio Carlos 6627, Belo Horizonte MG, Brazil
wlnico@mit.edu

***Abstract*—This is a theoretical study on how to address vector problems in electromagnetism by meshfree Galerkin methods. Oftentimes the field quantities are constrained by a divergence-free condition, which makes the numerical treatment via finite element procedures particularly challenging. Moreover, since the hallmark of true meshfree methods is the complete absence of any kind of mesh or grid, the methods rely on a very simple geometrical structure, for which special care needs to be taken to construct the function spaces. Then the use of a Lagrange multiplier in addition to the field quantities may provide correct solutions, if the underlying function spaces satisfy a certain condition.**

***Index Terms*—Computational electromagnetism, Inf-sup condition, Meshfree methods, Mixed formulations.**

I. Introduction

Meshfree (or meshless) methods gained some popularity in the 1990's as an alternative to the traditional finite element method (FEM). The use of these techniques in computational electromagnetism can be traced back to the early years of the past decade, and since then they have attracted the attention of researchers, working on topics such as static fields and wave propagation [1]. But the majority of works published thus far deals with scalar field problems, whereas the analysis of more realistic devices clearly requires vector field quantities.

The extension from scalar to vector problems presents numerous challenges [2]; the lack of mesh support prevents the easy construction of spaces of vector basis functions reflecting the topology of the electromagnetic quantities [3].

II. Overview

The idea is to introduce two spaces: one for the components of the vector field and another for the Lagrange multiplier. The method should be flexible to allow an easy construction of these spaces. The Method of Finite Spheres (MFS) shows a superior performance in this respect [4]. If the discretization requires N nodes, it employs a set $\{\varphi_I\}_{I=1}^{N}$ of partition of unity basis functions, enriched by local approximation bases. This gives rise to a two-parameter global approximation; a generic quantity w calculated at a point $\boldsymbol{x}$ is discretized as

$$w_h(\boldsymbol{x}) = \sum_{I=1}^{N} \sum_{m\in\mathcal{J}} \varphi_I(\boldsymbol{x}) p_m(\boldsymbol{x}) \alpha_{Im}. \quad (1)$$

$\mathcal{J}$ is an index set, p_m is a member of the local basis associated with node I and α_{Im} are the degrees of freedom.

We are interested in the discretized form of the vector wave equation; with the use of a (scalar) Lagrange multiplier p_h the governing equation is

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u}_h - k^2\boldsymbol{u}_h + \nabla p_h = \boldsymbol{f} \text{ in } \Omega \quad (2)$$
$$\nabla\cdot\boldsymbol{u}_h = 0 \text{ in } \Omega.$$

(Ω is the computational domain, $\boldsymbol{u}_h$ is the vector field quantity, k is the wavenumber and $\boldsymbol{f}$ is a source term.)

In essence, the concepts regarding mixed finite elements are adapted to the meshfree setting [5,6]. Different local bases [i.e., different choices of the p_m terms in (1)] are chosen for $\boldsymbol{u}_h$ and p_h; this yields global spaces $\mathbb{U}_h$ and $\mathbb{P}_h$ with different characteristics. One of the requirements (the other is the coercivity) for the existence and uniqueness of a weak solution of (2) is the *inf-sup condition*

$$\exists\beta_h > 0 \quad \inf_{q_h\in\mathbb{P}_h\backslash\{0\}} \sup_{\boldsymbol{v}_h\in\,\mathbb{U}_h\backslash\{\boldsymbol{0}\}} \frac{\int_\Omega q_h\,\nabla\cdot\boldsymbol{v}_h\,d\Omega}{\|\boldsymbol{v}_h\|_{\mathbb{U}_h}\|q_h\|_{\mathbb{P}_h}} \geq \beta_h. \quad (3)$$

Now that $\mathbb{U}_h$ and $\mathbb{P}_h$ are meshfree function spaces, a formal proof that they satisfy (3) is, if possible, not currently known. Hence we resort to a numerical test that has been used already for finite element discretizations in solid and fluid mechanics, and shell structural analyses [6,7,8,9].

If the two spaces pass the test, we say they form a *compatible pair*. In this work we consider various pairs of meshfree spaces and determine whether they are compatible. We find that the resulting global spaces $\mathbb{U}_h$ and $\mathbb{P}_h$ satisfy (3) with optimal convergence if the interpolation order used for the components of $\boldsymbol{u}_h$ is higher than that for p_h and is judiciously selected. In the paper, numerical solutions and convergence studies of the wave equation discretized by compatible pairs will be provided.

References

[1] W. Nicomedes, R. Mesquita and F. Moreira, 'The meshless local Petrov-Galerkin method in two-dimensional electromagnetic wave analysis," *IEEE Trans. Antennas & Prop*., v. 60, no 4, pp. 1957-1968, 2012.

[2] C. Lu and B. Shanker, "Generalized finite element method for vector electromagnetic problems," *IEEE Trans. Antennas & Prop*., v.55, no 5, pp.1369-1381, 2007.

[3] A. Bossavit, *Computational Electromagnetism*, 1st ed., Academic Press, 1997.

[4] S. De and K.J. Bathe, "The method of finite spheres," *Computational Mechanics*, v. 25, pp. 329-345, 2000.

[5] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Elements*, 1st ed., Springer Series in Computational Mathematics, 1991.

[6] K.J. Bathe, *Finite Element Procedures*, Prentice Hall, 1996.

[7] K.J. Bathe, "The inf-sup condition and its evaluation for mixed finite element methods," *Computers and Structures*, v. 59, pp. 243-252, 2001.

[8] S. De and K.J. Bathe, "Displacement/pressure mixed interpolation in the method of finite spheres," *Int. J. Numer. Meth. Engng*., v.51, pp.275-292, 2001.

[9] D. Chapelle and K.J. Bathe, *The Finite Element Analysis of Shells – Fundamentals*, Springer 2011.