A Meshless Local Boundary Integral Equation Method for Three Dimensional Scalar Problems

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Abstract— In this work we apply a meshless method based on Local Boundary Integral Equations (LBIE) to find the solution to boundary value problems. We discretize the weak form through the use of special basis functions that, unlike the Finite Element Method (FEM), are not confined to an element and do not need the support of an underlying mesh. The approach developed can be applied to general 3D scalar boundary value problems that arise in areas such as electrostatics and acoustic scattering, among others.

I. PRESENTATION AND OVERVIEW

In electromagnetics the use of meshless methods is still incipient. These methods comprise a family, in which each member has a particular feature that can be explored when applied to a given problem. Despite the fact of there being a number of different meshless methods, all share the same philosophy: in the region where the numerical solution is to be found, some points, called *nodes*, are spread and to each node a special function, called *shape function,* is associated. These shape functions do not have analytical expressions, demanding a numerical scheme to be constructed [1]. Besides that, the most important feature concerning these functions is that they must have a compact support, i.e., the region of space in which they are different from zero being called the node's *influence domain*. If a solution shall be found at a point $\vec{r}$, one has first to find out the nodes whose influence domains extend over the point in question. Then, the shape functions associated to these nodes are calculated and finally a weighted sum is performed:

$$u(\vec{r}) \sim u^h(\vec{r}) = \sum_{i=1}^{N} \phi_i(\vec{r})\hat{u}_i = \mathbf{\Phi}(\vec{r})\hat{\boldsymbol{u}} \qquad (1)$$

where ϕ_i is the i-node shape function and $\hat{u}_i$ is the associated *nodal parameter*. At this point becomes apparent the importance of the compact support property of the shape functions: only the closest nodes to $\vec{r}$ contribute to u^h. This feature is directly linked to the sparsity pattern of the final matrix. The size of the domains can be adjusted, but should be kept as small as possible, lest so many nodes influence $\vec{r}$, thus leading to more populated matrices.

II. THE LBIE METHOD APPLIED TO LAPLACE'S EQUATION

Let Γ be a spherical region with global boundary $\partial\Gamma$. Let u be the desired function, w a test function and $\vec{p}$ a given point where u shall be calculated. First, applying the weighted residual method to Laplace's equation, or equivalently, from Green's second identity for a pair of functions u and w, we substitute back $\nabla^2 u = 0$. For the LBIE method, w is chosen so that $\nabla^2 w = -\delta(\vec{r} - \vec{p})$, then $w(\vec{r}) = 1/(4\pi|\vec{r} - \vec{p}|)$. We get the global integral equation:

$$-\alpha(\vec{p})u(\vec{p}) = \oiint_{\partial\Gamma} u_D \frac{\partial w}{\partial n} dS - \oiint_{\partial\Gamma} w \left(\frac{\partial u}{\partial n}\right)_N dS \qquad (2)$$

where $\alpha = 1$ if $\vec{p} \in \Gamma$ and $\alpha = 0.5$ if $\vec{p} \in \partial\Gamma$, and u_D and $(\partial u/\partial n)_N$ are the prescribed Dirichlet and Neumann conditions, respectively. At this point a remarkable feature of LBIE becomes apparent: instead of performing the integrations in the global boundary $\partial\Gamma$, we carry them out at the boundary of a local spherical domain Ω centered at $\vec{p}$ [2]. In doing so, the local boundary conditions at $\partial\Omega$ are no longer the known prescribed ones. To avoid at least one of them, a new test function w^* is chosen so that it takes the value 0 at the local boundary $\partial\Omega$ (w^* is obtained from w through the subtraction of a *companion solution*). The boundary integral equation now has gone local:

$$-\alpha(\vec{p})u(\vec{p}) = \iint_{\partial\Omega\cap\Gamma} u \frac{\partial w^*}{\partial n} dS + \\ + \iint_{\Omega\cap\partial\Gamma} u_D \frac{\partial w^*}{\partial n} dS - \iint_{\Omega\cap\partial\Gamma} w^* \left(\frac{\partial u}{\partial n}\right)_N dS \qquad (3)$$

It should be noticed that if $\Omega \cap \partial\Gamma \neq \emptyset$, the region of integration is $\big((\partial\Omega \cap \Gamma) \cup (\Omega \cap \partial\Gamma)\big)$. Note that if $\Omega \cap \partial\Gamma = \emptyset$, then $\partial\Omega \cap \Gamma = \partial\Omega$, meaning that the prescribed Dirichlet and Neumann conditions do not need to be taken into account (the last two terms in (3)).

III. LBIE DISCRETIZATION AND PRELIMINARY RESULTS

The function u is substituted by a expansion in shape functions like (1). Then (3) is enforced at all nodes located at $\vec{p} = \vec{r}_i$, the integrations being performed at the boundary $\partial\Omega$ of small balls around each node i, or at the intersections between the small balls and the global boundary, if that is the case. After the process is finished, we end up with a linear system in $\hat{\boldsymbol{u}}$, which, when solved, allows the determination of u anywhere through (1). As an example, we applied the LBIE method to Laplace's equation in a unit sphere subject to the Dirichlet condition $u_D = 3cos^2\theta + 3cos\theta + 1$ on $\partial\Gamma$. Figure (1) shows the result along a path C defined in spherical coordinates as $r = 0.5$; $0 < \theta < 180°$; $\varphi = 0$.

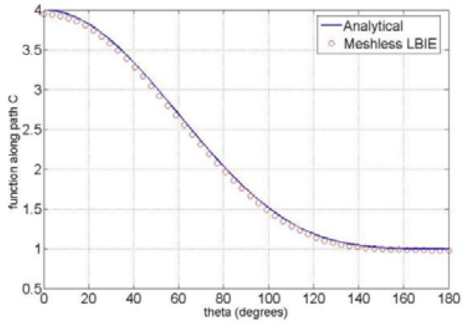

Fig. 1. Solution to Laplace's equation along a path C. The analytical result is $2(r/a)^2P_2(cos\theta) + 3(r/a)P_1(cos\theta) + 2P_0(cos\theta)$.

IV. REFERENCES

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