Piecewise affine models of chaotic attractors: The Rössler and Lorenz systems

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This paper proposes a procedure by which it is possible to synthesize Rössler [Phys. Lett. A 57, 397–398 (1976)] and Lorenz [J. Atmos. Sci. 20, 130–141 (1963)] dynamics by means of only two affine linear systems and an abrupt switching law. Comparison of different (valid) switching laws suggests that parameters of such a law behave as codimension one bifurcation parameters that can be changed to produce various dynamical regimes equivalent to those observed with the original systems. Topological analysis is used to characterize the resulting attractors and to compare them with the original attractors. The paper provides guidelines that are helpful to synthesize other chaotic dynamics by means of switching affine linear systems. © 2006 American Institute of Physics. [DOI: 10.1063/1.2149527]

Chaotic systems...
of “neighborhoods” when compared to traditional piecewise models\textsuperscript{14,10} and to more recent methods.\textsuperscript{16}

We will therefore focus our attention on the ability to build piecewise models by associating one affine subsystem to each fixed point, the subsystems being linked by switching surfaces. Moreover, an emphasis will be given to the choice of the switching surface in terms of attractor features, which greatly contrast with the other methods where, very often, the submodels are distributed over a regular grid.\textsuperscript{16} This brings model structures which are radically different from those resulting from other procedures.

This paper illustrates this new procedure by which it is possible to estimate different chaotic dynamics as those induced by two mechanisms, which are folding and tearing for the Rössler and Lorenz systems, respectively. Topological analysis\textsuperscript{17–19} is used to characterize the resulting attractors and to compare them with the original attractors. Unlike other procedures where a close approximation of the equilibrium manifolds is intended and achieved,\textsuperscript{16} topological analysis is concerned to check how the unstable periodic orbits, embedded within the attractor, globally relate to each other.

This work is therefore a first step toward a general methodology to produce nonlinear dynamics with affine linear components, which could turn out to be greatly desirable in the construction of hardware systems that should have nonlinear dynamics of a certain type.

This paper is organized as follows. In Sec. II the main motivation and ideas are provided considering the case of the Rössler system. Section III briefly reviews some key concepts on topological analysis of chaotic attractors. Such concepts will turn out to be important to choose valid switching surfaces. Section IV describes the piecewise affine representation being proposed and how to use it to represent Rössler and Lorenz dynamics. That section also will perform topological analysis of the resulting attractors in order to provide some insight into the whole model-building procedure. Finally, the main conclusions are presented in Sec. V.

II. FIXED POINTS AND AFFINE SUBSYSTEMS

In 1963, using a simplified model of Rayleigh-Bènard convection, Lorenz was among the first to describe a system that presented continuous chaos.\textsuperscript{9} The so-called Lorenz system,

\begin{align}
\dot{x} &= -sx + sy, \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= -bz + xy,
\end{align}

depends on parameters \((s, r, b)\). The fixed points of the Lorenz system are

\[
\mathbf{p}_r = \begin{bmatrix}
x_+ = \sqrt{b(r-1)} \\
y_+ = \sqrt{b(r-1)}, \\
z_+ = r - 1
\end{bmatrix}, \quad \mathbf{p}_0 = \begin{bmatrix}
x_0 = 0 \\
y_0 = 0, \quad \text{and} \\
z_0 = 0
\end{bmatrix}
\]

Thus, for all the parameter values here investigated, the Lorenz system has two unstable foci \((\mathbf{p}_r)\) connected to each other by a saddle point \((\mathbf{p}_0)\). Such fixed points induce two “glued” spirals which form the attractor.

From Poincaré’s works, it is known that phase portraits are structured around fixed points. Using bounding tori, Tsankov and Gilmore showed that there is a significant difference in the role played by the foci and the saddle-fixed points.\textsuperscript{15} In particular, it is not possible to have two focus-fixed points surrounded by the flow unless they are separated by a saddle-fixed point, also surrounded by the flow (Fig. 1). The bounding torus shown in Fig. 1 corresponds to the Lorenz system with tearing mechanism. In fact, the role played by the saddle-fixed point is to allow the transition from one spiral to the other, that is, from the influence domain of one focus to the other.

As will become clear later in the paper, it will be convenient to see that the saddle point in the Lorenz system plays the role of a switching surface. Such a surface would then determine when the system is under the influence of one focus-fixed point, or of the other. In the Lorenz system, only focus-fixed points will be associated with affine subsystems, as will be detailed in Sec. IV B.

In order to have a simplest chaotic attractor—without symmetry property—the Rössler system was proposed as a “model of the Lorenz system.” It is described by the following equations:\textsuperscript{1}

\begin{align}
\dot{x} &= -y - z, \\
\dot{y} &= x + ay, \\
\dot{z} &= b + (x - c)z,
\end{align}

where \(a, b,\) and \(c\) are constants. This is simpler than the Lorenz system because it only presents a single spiral. The Rössler has two fixed points:
and one decreasing branch splitted by one critical point, it

These two fixed points are two saddle foci. Fixed point

has an unstable manifold \( E^u \subset \mathbb{R}^2 \) associated with a divergent spiral—mainly responsible for the stretching—and a stable manifold \( E^s \subset \mathbb{R} \). The second fixed point, \( p_\pm \), has a stable manifold \( E_s \subset \mathbb{R}^2 \)—a convergent spiral—and an unstable manifold \( E_u \subset \mathbb{R} \). From the bounding torus point of view, both are focus-fixed points but only \( p_- \) is surrounded by the flow (Fig. 2). Thus, there is no need to have a saddle-fixed point to connect one to the other, as required for the Lorenz system. Nevertheless, both fixed points have their corresponding domains of influence. Therefore as the system evolves in phase space, it can be interpreted as smoothly “switching” from the influence domain of one fixed point to the domain of the other. One of the goals of this paper is to mimic the Rössler system by abruptly switching from one domain to the other. It will be necessary to define the domains and also the switching law. This will be detailed in Sec. IV A.

In the Rössler system, the switching is induced by the nonlinearity which acts when the trajectory is sufficiently far from fixed point \( p_- \), that is, beyond the threshold \( x-c \) in the third equation of the Rössler system (2). In fact, the nonlinearity acts when the trajectory is sufficiently close to fixed points \( p_\pm \) where its converging spiral induces the folding by sending the trajectory back to the neighborhood of fixed point \( p_- \) along its unstable manifold \( E^u \). Thus, fixed point \( p_- \) is mainly responsible for the stretching and fixed point \( p_+ \) for the folding.

In fact, as soon as \( p_- \) becomes influential on the trajectory, a folding occurs. \( p_- \) only induces a stretching mechanism. These two domains are thus associated with different topological properties. For instance, when the chaotic attractor is characterized by a unimodal map with one increasing branch and one decreasing branch splitted by one critical point, it

will be shown in Sec. III that the increasing branch corresponds to the domain of influence of \( p_- \) and the decreasing branch to the domain of \( p_+ \).

The main idea of this paper is to associate linear dynamics—an affine subsystem—to each domain of influence of each focus-fixed point, the subsystems being linked by switching laws that determine when and how the trajectory passes the transition regions. To validate the piecewise affine subsystems, some concepts of topological analysis, as described in the next section, will be used.

### III. TOPOLOGICAL ANALYSIS

This section is devoted to a topological analysis of phase portraits. Such an analysis starts by computing a first-return map to the Poincaré section of the phase portrait [Fig. 3(a)]. For the Rössler system, it may be defined as

\[
P = \{ (y_n,z_n) \in \mathbb{R}^2 \mid x_n = 0, x_n < 0 \}.
\]

The first-return map [Fig. 3(b)] is made of two monotonic branches separated by a critical point located at the maximum. The first-return map induces therefore a partition of the phase portrait in two branches [Fig. 3(c)]. A symbol is associated with each branch. Chaotic trajectories and the periodic orbits constituting their skeleton can thus be encoded over the symbol set \{0, 1\}. The symbol “0” is associated with the increasing branch and the symbol “1” with the decreasing branch. Periodic orbits may thus be encoded by symbolic strings. For instance, a period-2 orbit having one intersection with the Poincaré section located on the branch 0 and one located in the branch 1 is designated by the sequence (10). A period-3 orbit would have three symbols, and so on.

In a general way, increasing branches are preserving order and decreasing branches are reversing order. From the first-return map shown in Fig. 3(b), it appears that the phase portrait can be divided in one preserving order branch and one reversing order branch. A preserving order branch represents an even number of half-turns while a reversing order branch represents an odd number of half-turns. In the case of the Rössler system, the corresponding template is shown in Fig. 3(c). An adequate template must predict topological invariants such as linking numbers between pairs of periodic orbits. Periodic orbits embedded within the attractor can be approximated by segments of the chaotic time series that mimic the behavior of nearby unstable periodic orbits. A “close return” method\(^{18}\) is applied to the Poincaré section to extract them. Periodic orbits can be viewed as knots and the template as a “knot holder.” The template synthesizes all the topological properties of the chaotic attractor, that is, the relative organization of all the periodic orbits embedded within the attractor. One of the very useful topological invariants—a quantity preserved under an isotropy which is a continuous deformation without any cutting—is the linking number defined as follows (Fig. 4).

Let \( \alpha \) and \( \beta \) be two knots defining a link \( L \subset \mathbb{R}^3 \). Let \( \sigma \) denotes the set of crossings of \( \alpha \) with \( \beta \). Then the linking number reads

\[
\chi(\alpha, \beta) = \begin{cases} +1 & \text{if } \sigma \text{ is an even number of crossings} \\ -1 & \text{if } \sigma \text{ is an odd number of crossings} \\ 0 & \text{if } \sigma = 0 \end{cases}
\]
where $\epsilon$ is the sign of each crossing $p$ with the usual convention (see Fig. 4).

The linking number $lk(\alpha, \beta)$ of two periodic orbits $\alpha$ and $\beta$ is the half of the algebraic sum of all crossings between $\alpha$ and $\beta$ (ignoring self-crossings). A template is validated when all linking numbers it predicts are equal to those computed from the orbits extracted from the attractor. The template shown in Fig. 3(c) has been validated in Ref. 18. Further details for such a topological characterization procedure are extensively discussed in Refs. 18 and 19.

A template remains valid over a certain range of the parameter values for which the number of monotonic branches in the first-return map does not change. Nevertheless, the population of periodic orbits may change through some bifurcations. For instance, in the Rössler system, when the bifurcation parameter $a$ is increased, new periodic orbits are created and the chaotic attractor increases in size. The corresponding first-return map is constituted by more than two branches and, for $a=0.556$, up to 11 monotonous branches may be identified. All the additional branches involve a folding mechanism and are under the main influence of

\[
lk(\alpha, \beta) = \frac{1}{2} \sum_{p} \epsilon(p),
\]

FIG. 3. (a) Chaotic attractor solution to the Rössler system. (b) First-return map to a Poincaré section. (c) Template of the attractor. Parameter values: $a=0.398$, $b=2$, and $c=4$. The partition of the attractor and the four switching surfaces are shown in (a).

FIG. 4. Convention of positive and negative crossings.
of $p_n$. The corresponding attractor is designated as the screw attractor.\(^{21}\) For $a$ greater than 0.556, there is metastable chaos, that is, the trajectory visits the neighborhood of the unstable periodic orbit solution to the Rössler attractor before being ejected to infinity.\(^{18}\) The dynamics of the Rössler system can therefore be investigated for $a<0.556$, $b$ and $c$ remaining constant.

A bifurcation diagram synthesizes the evolution of the dynamics under the change of the bifurcation parameter $a$ (Fig. 5). It is built as saving a large number (e.g., 100) intersections of the trajectory with the Poincaré section for each value of the parameter. In such a diagram, one may identify a period-doubling cascade between $a=0.2$ and $a=0.385$, then a chaotic solution is observed. The chaotic solution arises when the orbit with period $2^n$ with $n\to\infty$ has been created.

Increasing parameter $a$ further, various periodic windows are observed. In each of them, a new period-doubling cascade can be seen. Orbits have period equal to $2^n p$, where $p$ is the period of the orbit at the left of the window. For instance, a period-3 window is observed around $a=0.410$. As long as the first-return map is unimodal, that is, has two monotonic branches with a differentiable maximum, the sequence of bifurcation can be predicted using the so-called unimodal order.\(^{16}\) In this case, the sequence of periodic orbit occurrence can be predicted. Thus, when the $a$ parameter of the Rössler system is varied from 0.125—the Hopf bifurcation leading to the limit cycle encoded by (1)—to 0.43295 for which the symbolic dynamics over symbols 0 and 1 is complete—any symbolic sequence corresponds to a periodic orbit within the attractor—the orbits occur in the attractor according to the unimodal order. Orbits with period less than 7 are reported in Table I according to the unimodal order. Thus, knowing the last created orbit is sufficient to determine the population of periodic orbits. The sequence of the last created orbits is called the kneading sequence. For instance, the Rössler system has a population of periodic orbits defined by the kneading sequence (10110), that is, all the periodic orbits created before the orbit (10110) are embedded within the attractor.

The topology of the Lorenz system is slightly more complicated than those of the Rössler system. As recently shown,\(^{22,23}\) depending on the parameter values, two different mechanisms can be identified in the Lorenz system. The most common is the tearing induced by the surrounded saddle-fixed points [Fig. 6(a)]. Tearing “tears” the flow such that infinitesimally close trajectories can “split” into different branch lines which do not remain close.\(^{23}\) This is the mechanism which is involved in the usual Lorenz attractor. Tearing has a very definite signature in a first-return map to a Poincaré section: the existence of a nondifferentiable point in the map. This is the cusp observed at the maximum of the so-called “Lorenz map.” The second mechanism is the folding which corresponds to what is observed in the Rössler system: a typical folding is characterized by a differentiable unimodal first-return map, that is, one increasing branch separated from one decreasing branch by one differentiable maximum. As explained in Ref. 23, when the Lorenz attractor can be embedded within a genus-1 bounding torus [Fig. 6(b)], only folding mechanism is present. As soon as a saddle-fixed point is surrounded by the flow, the attractor is embedded within a genus-3 bounding torus and tearing mechanism will be present. In fact, the Lorenz genus-1 bounding torus can be seen as a genus-3 bounding torus where the trajectory cannot visit the domain between the

\begin{table}[h]
\centering
\caption{Sequence of periodic orbits ordered according to the unimodal order. Only orbits with a period smaller than 7 are reported.}
\begin{tabular}{lll}
\hline
Symbolic sequence & & \\
(1) & (100) & (100010) \\
(10) & (100101) & (100011) \\
(101) & (10010) & (1001) \\
(1011) & (10011) & (10000) \\
(10111) & (100111) & (100001) \\
(101111) & (100110) & (100000) \\
(1011111) & (100101) & (1000) \\
(10111111) & (100100) & (100) \\
\hline
\end{tabular}
\end{table}
saddle and the foci (see Ref. 23 for details). Thus, varying some parameters of the Lorenz system leads to different topologies as characterized by the two templates shown in Fig. 6.

**IV. PIECEWISE AFFINE MODEL BUILDING**

Section II has provided motivation to search for a model-building procedure by which the system is switched from one affine subsystem to another. As previously explained, the number of affine subsystems is related to the number of focus-fixed points. What remains to do is to determine the switching law. The present section aims at shedding some light on this point. As mentioned before, the definition of an affine systems is \( \dot{x} = Ax + b \), where \( A \) and \( b \) are constants. Clearly, such an affine system with respect to the origin \( x=0 \) can be considered as a linear system centered at some point \( x=(A+f)\bar{b} \).

Therefore, the structure of a piecewise affine model can be described as

\[
\dot{x} = \sum_{i=1}^{m} f_i[s(x)] A_i(x - p_i),
\]

where \( x \in \mathbb{R}^n \) is the state vector, \( m \) is the number of affine subsystems, and \( p_i \in \mathbb{R}^n \) is the fixed point to which the affine subsystem is associated. Constant matrices \( A_i \in \mathbb{R}^{n \times n} \) define the local linear dynamics of the affine subsystems and \( s(x) \) is the switching surface between the domains where the subsystems are active and \( f_i[\cdot] \) is a Boolean function.

The functions \( f_i \) are designed in such a way that, at any given time, only one of them equals unity.

Equation (6) describes a system composed of \( m \) affine subsystems centered at the corresponding fixed points \( p_i (i = 1, 2, \ldots, m) \) and governed by the (local) linear dynamics described by the corresponding matrix \( A_i \). Which of these subsystems are active at time \( t \) is determined by the switching law (surface) \( s(x) \) and the functions \( f_i[\cdot] \).

Building a piecewise affine model of form (6) consists of determining (i) the fixed points \( p_i \), (ii) the number \( m \) of affine subsystems (space partitions), (iii) the dynamic matrices \( A_i \), and (iv) the switching law \( s(x) \). The choice of the functions \( f_i[\cdot] \) is an additional ingredient. For instance, it can be chosen such that the transition from one subsystem to another is smooth rather than abrupt (switched), as indicated in (7). Nonetheless, throughout this paper \( f_i[\cdot] \) will be taken, as shown in (7).

**A. The case of the Rössler system**

As seen in Sec. I the Rössler dynamics can be understood (and described) by two affine subsystems and a switching mechanism. Such affine subsystems can be obtained by linearizing the dynamics around the two fixed points of the Rössler attractor. A model under form (6) can thus be written as

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = f_i[s(x)] \begin{bmatrix}
0 & -1 & -1 \\
1 & 0.398 & 0 \\
0 & (x_+ - 4) & 0
\end{bmatrix} \begin{bmatrix}
x - x_- \\
y - y_- \\
z - z_
\end{bmatrix}
+ f_2[s(x)] \begin{bmatrix}
0 & -1 & -1 \\
1 & 0.398 & 0 \\
0 & (x_+ - 4) & 0
\end{bmatrix} \begin{bmatrix}
x - x_+ \\
y - y_+ \\
z - z_+
\end{bmatrix}.
\]  

Note that in this first attempt to build affine piecewise models, the model obtained are only valid for given parameter values. The two square matrices in Eq. (8) are the Jacobian matrices of the Rössler system evaluated at \( p_- \) and \( p_+ \), respectively.

Model (8) is piecewise affine and, because a single function \( f_i \) equals one at a time, the system evolves according to a single subsystem at any given time. Nevertheless, the overall dynamics is determined by the whole set of subsystems and how the trajectory switches from one affine subsystem to the other. Therefore, one of the very relevant ingredients for the global dynamics is the switching surface \( s(x) \).

In the case of the Rössler system, the switching surface can be determined based on topological properties of the attractor. As seen in Sec. II, the first-return map of the Rössler attractor has two branches, each of them being associated with one branch of the template. Since folding—the influence of \( p_+ \)—only occurs in branch 1, it seems natural to use the subsystem associated with fixed point \( p_- \) for branch 1 and the subsystem associated with fixed point \( p_+ \) for branch 0.
To investigate the influence of the switching mechanism on the resulting dynamics, the following four different switching surfaces will be considered:

\[ s_1(x) = \{ x \in \mathbb{R}^3 | x = 3.59 \} \]

\[ s_2(x) = \{ x \in \mathbb{R}^3 | y = -3x + 10.6 \} \]

\[ s_3(x) = \begin{cases} 
  \{ x \in \mathbb{R}^3 | x + 0.7y + 4.5 = 0 \}, & \text{if } y \geq -1.482 \\
  \{ x \in \mathbb{R}^3 | x + 2.35y - 0.02 = 0 \}, & \text{if } y < -1.482 
\end{cases} \]

\[ s_4(x) = \{ x \in \mathbb{R}^3 | 0.95y^2 + 3y - 6 - x = 0 \} \]

where \( x = [x \ y \ z]^T \). All these surfaces are located around the threshold beyond which the nonlinearity is active. The main differences between these surfaces are in their shapes [Fig. 3(a)] and in the volume of the phase space where the subsystem associated with fixed point \( p_+ \) is active. An important point for the choice of the switching surfaces is that the transition between the subsystem associated with point \( p_+ \)—hereafter referred to as subsystem \( p_+ \)—and the subsystem associated with the fixed point \( p_- \)—hereafter referred to as subsystem \( p_- \)—must be located in the neighborhood of the projection of the maximum of the \( z \) variable in the \( x-y \) plane [Fig. 3(a)]. This corresponds in the Rössler system to the point where the nonlinearity saturates and where the dynamics is dominated by the stable manifold \( E^n \) of fixed point \( p_- \).

At first sight, the transition between subsystem \( p_- \) and subsystem \( p_+ \) occurs near the partition of the attractor or, at least, in branch 1. This transition is significantly influent on the dynamics. The resulting chaotic attractors of the four different piecewise affine systems are shown in Fig. 7. These

FIG. 7. Chaotic attractor solutions for the four piecewise affine models. The switching surface is plotted using dashed lines for each model. The first-return maps to a Poincaré section are also computed.
four attractors are of two types. First, attractors corresponding to surface $s_3(x)$ and $s_4(x)$ are very close to the original Rössler attractor [compare Figs. 7(c) and 7(d) with Fig. 3(a)]. Their first-return maps are both unimodal with a differentiable maximum. Model $M[s_3(x)]$ has an increasing branch slightly more developed than in the original Rössler attractor [compare map in Fig. 7(d) with the map in Fig. 3(b)]. This is confirmed by the kneading sequence which is $100101$ rather than $101111$ for the original Rössler dynamics and for model $M[s_4(x)]$.

Linking numbers computed from periodic orbits extracted from these two attractors are in agreement with those predicted by the template shown in Fig. 3(c). Consequently, attractors of models $M[s_3(x)]$ and $M[s_4(x)]$ are topologically equivalent to the original Rössler attractor [Fig. 3(a)]. The most accurate model is model $M[s_3(x)]$ which has the same population of periodic orbits as that of the original attractor. Model $M[s_4(x)]$ has a population of periodic orbits which is slightly more developed than that of the Rössler system for the parameter values considered. However, the population of periodic orbits of $M[s_4(x)]$ does correspond to the counterpart population in the Rössler system with slightly different parameter values.

On the other hand, models $M[s_1(x)]$ and $M[s_2(x)]$ produce attractors which are very different in their topologies. As shown in Figs. 7(a) and 7(b), the trajectory may turn twice per cycle around fixed point $p_-$. This means that there are two “foldings.” The first of them is, in fact, a global torsion by two half-turns [right bottom part of the attractors shown in Figs. 7(a) and 7(b)] and the second folding is a folding in the usual sense, that is, one of the two branches is folded over the other. The template for these attractors is therefore different than those shown in Fig. 3(c) and a global torsion must be added. This is shown in Figs. 8(a) and 8(b).

The corresponding first-return maps are still unimodal with a differentiable maximum. It can be shown that the two
affine model associated with the inner fixed point to the second affine model, which corresponds to the outer fixed point. In order to illustrate that assertion, a bifurcation diagram was computed using $\alpha$ as a bifurcation parameter. Parameter $\beta$ was chosen in such a way as to keep unchanged the point at which the two branches of $s_i(x)$ encounter.

Decreasing $|\alpha|$ is equivalent to increasing the angle between the two branches of the switching surface. The bifurcation diagram shown in Fig. 9 shows a “blurred” inverse period-doubling cascade from $|\alpha| > 2.5$. This means that when the angle between the two branches of the switching surface is decreased, the dynamics is more constrained in the sense that the time for the system to evolve from one such branch to the other is insufficient to develop a full folding. Contrary to this, increasing the angle—which is equivalent to decreasing $|\alpha|$—provides sufficient time for the system to experience additional foldings. The result of this process is a first-return map which may have up to five monotonic branches, as shown in Fig. 10 where $\alpha = -0.6$. With this low value of $|\alpha|$, the domain, over which the affine model associated with the outer fixed point is active, is larger than for $\alpha = -2.35$, as shown in Fig. 7(c). Thus, for $\alpha = -0.6$ successive foldings occur before the trajectory crosses the upper branch of the switching surface. This shows that multimodal chaotic attractors can also be obtained using piecewise affine models. Note that $\alpha$ can be tuned to recover any desired population of periodic orbits of the original dynamics.

**B. The case of the Lorenz system**

As discussed in Sec. III, the fixed-point distribution of the Lorenz system is different from those of the Rössler system. In particular, the presence of the saddle leads to a tearing mechanism (never observed in the Rössler system). Moreover, the Lorenz system is invariant under a rotation symmetry. The dynamics underlying the Lorenz system is therefore very different from the Rössler system. Applying our procedure to build a piecewise affine model is not so straightforward. According to Ref. 22, it is possible to clearly distinguish the domains of the parameter space of the Lorenz system where only tearing or folding occurs by using the parameter transformation

$$ s = s_0 + \rho(s_1 + s_0), $$

$$ r = r_0 + \rho(r_1 + r_0), $$

$$ b = b_0 + \rho(b_1 + b_0), $$

with $(s_0, r_0, b_0) = (10, 28, \frac{8}{3})$ and $(s_1, r_1, b_1) = (30, 278.56, 1)$. Thus, the Lorenz system only presents a tearing for $\rho < 0.15$ and only a folding for $\rho > 1.0$. The attractor presents tearing and folding for $0.15 < \rho < 1.0$.

Our aim is to build a piecewise affine model of the Lorenz system with the usual attractor, that is, with $(s_0, r_0, b_0)$. For these parameter values, $\rho = 0$, and only a tearing mechanism occurs. The fixed-point coordinates of the Lorenz system are thus given by
Using the Jacobian matrices estimated at the two foci $p_\pm$, the simple piecewise affine model is proposed as

$$
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = f_1(s(x)) + f_2(s(x))
\begin{bmatrix}
3\sqrt{8} & 0 & -3\sqrt{8} \\
3\sqrt{8} & 1 & -3\sqrt{8} \\
3\sqrt{8} & 3\sqrt{8} & -\frac{8}{3}
\end{bmatrix}
\begin{bmatrix}
x - 3\sqrt{8} \\
y - 3\sqrt{8} \\
z - 27
\end{bmatrix}
\times
\begin{bmatrix}
-10 & 10 & 0 \\
-1 & -1 & +3\sqrt{8} \\
-3\sqrt{8} & -3\sqrt{8} & -\frac{8}{3}
\end{bmatrix}
\begin{bmatrix}
x + 3\sqrt{8} \\
y + 3\sqrt{8} \\
z - 27
\end{bmatrix}.
$$

(14)

Note that each affine subsystem can be mapped to the other under the rotation symmetry of the Lorenz system, that is, under the map $(x, y, z) \mapsto (-x, -y, z)$. As explained in Sec. II, it is natural (from a topological point of view) to replace the saddle with the switching surface. Since the Lorenz system is invariant under a rotation symmetry around the $z$ axis, the switching surface must contain the $z$ axis to preserve this symmetry. We thus consider the surface defined as

$$
s_5(x, \theta) = \begin{cases}
(x, \theta) \in \mathbb{R}^3 \times [0, \pi] | x = 0 & \text{if } \theta = \pi/2 \\
(x, \theta) \in \mathbb{R}^3 \times [0, \pi] | x - (1/\tan \theta)y = 0 & \text{if } \theta \neq \pi/2,
\end{cases}
$$

(15)

where parameter $\theta$ allows us to rotate the switching surface $s_5(x, \theta)$ around the $z$ axis. As shown in Fig. 11(b), parameter $\theta$ plays the role of a bifurcation parameter and when the bifurcation diagram is plotted versus $\eta = (1/\tan \theta)$, the sequence of bifurcation is similar to those observed in the bifurcation diagram of the Lorenz system plotted versus $\rho$ [Fig. 11(a)]. In particular, at the right part of the diagrams, there are inverse period-doubling cascades, a strong signature of an attractor with only a folding mechanism.23

The Lorenz dynamics with tearing mechanism is observed in the left part of the bifurcation diagrams. Thus, the piecewise affine model (14) has a unimodal map with a cusp

![FIG. 9. Bifurcation diagram of the affine piecewise model $M_s(x)$ vs $|a|$.](Image)

![FIG. 10. First-return map to a Poincaré section for the affine piecewise model $M([s(x)])$ with $\alpha = 0.6$.](Image)
when $\theta \approx 1.346 \ (1/\tan \theta \approx 0.229)$. The corresponding attractor is shown in Fig. 12(a) with a first-return map to the Poincaré section

$$P_L = \{(y_n, z_n) \in \mathbb{R}^2 | x_n = x_+ \hat{x}_n > 0 \} \cup \{(y_n, z_n) \in \mathbb{R}^2 | x_n = x_- \hat{x}_n < 0 \}.$$  \tag{16}

The map is computed using the $z$ variable which does not distinguish the two components of the Poincaré section (see Refs. 15 and 25 for details). When periodic orbits embedded within the attractor solution to model (14), the population is found to be very close to those extracted from the original Lorenz attractor with parameters $(s_0, r_0, b_0)$ and all linking numbers are equal to those predicted by the template shown in Fig. 6(a). The chaotic attractor solution to model (14) is therefore topologically equivalent to the Lorenz attractor. It is thus possible to reproduce the Lorenz dynamics using a piecewise affine model.

Varying $\theta$ allows us to vary a bifurcation parameter and the attractor has a different topology. When $\theta \approx 1.004 \ (1/\tan \theta \approx 0.636)$, the attractor is characterized by a unimodal first-return map with a differentiable maximum, that is, only folding a mechanism is involved [Fig. 12(b)]. The attractor solutions to model (14) with $\theta \approx 1.004$ are topologically equivalent to attractor solutions to the Lorenz system with $\rho > 1.0$. Varying $\theta$ allows us to adjust the population of unstable periodic orbits. Since parameter $(s, r, b)$ correspond to the Lorenz attractor with tearing, $\theta$ is actually a bifurcation parameter to adjust. As suggested by the bifurcation diagram shown in Fig. 11(b), an attractor with tearing and fold-
FIG. 12. Attractor projections. The left column corresponds to pure tearing $\theta = 1.346$ and the right column corresponds to pure folding $\theta = 1.004$. The bottom plots show the first-return maps to the Poincaré section $P_L$. 
ing mechanisms can be obtained with $1.004 < \theta < 1.396$. Parameter $\theta$ must, indeed, be considered as an additional bifurcation parameter.

V. CONCLUSION

This paper has developed a procedure to construct piecewise affine models with abrupt switching that are topologically equivalent to original systems described by a set of continuous ordinary differential equations. The procedure is presented for two different systems, namely, the Rössler and Lorenz systems.

The main framework consists of the following steps:

1. define the number and location of fixed points around which the dynamics will be organized,
2. evaluate the Jacobian matrix at each of the fixed points of the focus type defined in the previous step,
3. determine a switching surface based on topological guidelines and/or the relative organization of the fixed points (see text), and
4. simulate the piecewise affine model and compare the resulting attractor by means of topological analysis. The position of the switching surface determined in the previous step can be used for fine tuning.

In both cases investigated, only two fixed points were used for building the piecewise affine models. It seems that whenever a fixed point of the original system is required to split the domains of influence of two focus-fixed points surrounded by the flow—as in the case of the Lorenz system—such a fixed point is replaced in the piecewise affine model by a switching surface.

In examples provided in this paper, the attractors were always topologically equivalent to counterpart attractors of the original systems. This comes as no surprise because the affine models were constructed from the original Jacobian matrix and fixed points, which were assumed known. Following Poincaré’s reasoning, since in this case the dynamics are structured around fixed points and are determined by the local respective eigenstructures, the overall dynamics cannot differ drastically from that of the original system. However, if the affine models and/or the fixed points are not known it can be expected that the performance of the piecewise affine model will degrade.

It has been shown that a very critical ingredient is the choice of the switching surfaces since the topology of the chaotic attractor, which is the solution to the piecewise affine model, may be strongly affected by the nature of the switching surface. In fact, this paper has provided numerical evidences that show that it is possible to parametrize the switching surface in such a way that a given parameter works as a bifurcation parameter. As a consequence of this, different dynamical regimes—all of them topologically consistent with the original dynamics—are possible to obtain, including multimodal chaos. In the case of the Rössler system, such a multimodal chaos is characterized by the possibility to obtain screw attractor, that is, an attractor with many folding (strips of the template with various numbers of half-turns) during one revolution over the attractor. In the Lorenz case, multimodal chaos is obtained whenever the attractor includes both tearing and folding.

After this stage in piecewise model building of chaotic attractors it seems that the next steps are (i) to check the present procedure with other systems, (ii) to check with higher-order symmetries that our conjecture saddle-fixed points can always be replaced with switching surfaces, and (iii) to develop algorithms to construct piecewise affine models from data.

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